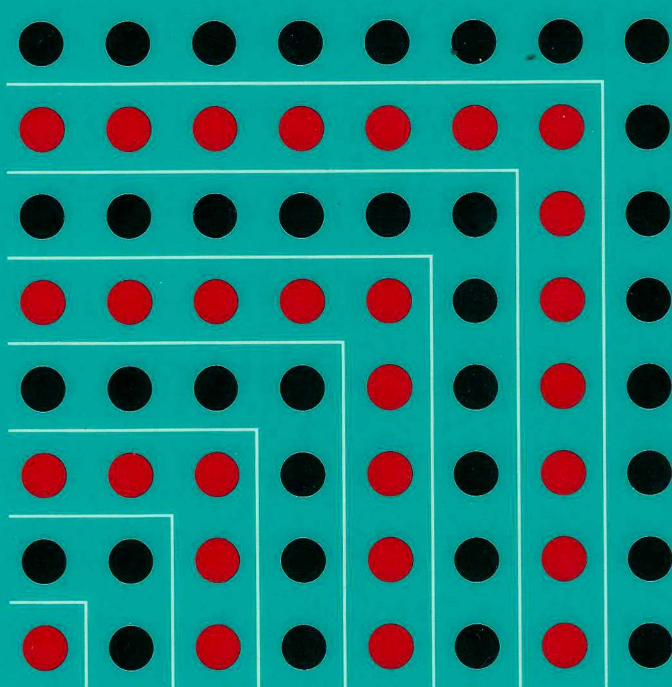
PROOFS WITHOUT WORDS

EXERCISES IN VISUAL THINKING



$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

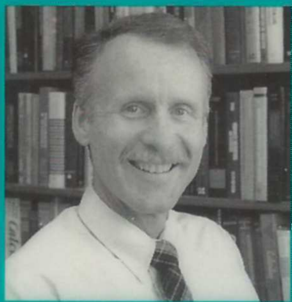
ROGER B. NELSEN

CLASSROOM RESOURCE MATERIALS / NUMBER 1

THE MATHEMATICAL ASSOCIATION OF AMERICA

PROOFS WITHOUT WORDS

EXERCISES IN VISUAL THINKING



Roger Nelsen received his Ph.D. in Mathematics from Duke University. Since 1969 he has taught at Lewis and Clark College, in Portland Oregon, where he is a professor of mathematics. He is currently an Associate Editor of the "Problems and Solutions" section of the *College Mathematics Journal.* His expository and research papers in mathematics have appeared in the *American Mathematical Monthly, Mathematics Magazine, The College Mathematics Journal, Nature, Journal of Applied Probability, Communications in Statistics, Probability Theory and Related Fields, Statistics and Probability Letters, Sankhyā,* and the *Journal of Nonparametric Statistics.* His main research interests are in probability and mathematical statistics.

Just what are "proofs without words?" First of all, most mathematicians would agree that they certainly are not "proofs" in the formal sense. Indeed, the question does not have a simple answer. But, as you will see in this book, proofs without words are generally pictures or diagrams that help the reader see *why* a particular mathematical statement is true, and also to see *how* one could begin to go about proving it true. While in some proofs without words an equation or two may appear to help guide that process, the emphasis is clearly on providing *visual* clues to stimulate mathematical thought. Proofs without words bear witness to the observation that often in the English language to see means to *understand*, as in "to see the point of an argument."

Proofs without words have a long history. In this collection you will find modern renditions of proofs without words from ancient China, classical Greece, twelfth-century India—even one based on a published proof by a former President of the United States! However, most of the proofs are relatively more recent creations, and many are taken from the pages of MAA journals.

The proofs in this collection are arranged by topic into six chapters: Geometry and Algebra; Trigonometry, Calculus and Analytic Geometry; Inequalities; Integer Sums; Sequences and Series, and Miscellaneous. Teachers will find that many of the proofs without words in this collection are well suited for classroom discussion and for helping students to think visually in mathematics.

Proofs Without Words

Exercises in Visual Thinking

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Classroom Resource Materials
Number 1

Proofs Without Words

Exercises in Visual Thinking

Roger B. Nelsen
Lewis and Clark College

Published and Distributed by

THE MATHEMATICAL ASSOCIATION OF AMERICA

Library of Congress Catalog Card Number 93-86388

ISBN 0-88385-700-6

Printed in the United States of America

Current Printing (last digit):
10 9 8 7 6 5 4 3 2

Introduction

see (sē) *v.*, **saw, seen, seeing.** —*v.t.*

. . .

5. to perceive (things) mentally; discern; understand: *to see the point of an argument.*

. . .

—THE RANDOM HOUSE DICTIONARY OF THE ENGLISH LANGUAGE (2nd ED.) UNABRIDGED.

"Proofs without words" (PWWs) have become regular features in the journals published by the Mathematical Association of America — notably *Mathematics Magazine* and *The College Mathematics Journal.* PWWs began to appear in *Mathematics Magazine* about 1975, and, in an editors' note in the January 1976 issue of the *Magazine,* J. Arthur Seebach and Lynn Arthur Steen encouraged further contributions of PWWs to the *Magazine.* Although originally solicited for "use as end-of-article fillers," the editors went on to ask "What could be better for this purpose than a pleasing illustration that made an important mathematical point?"

A few years earlier Martin Gardner, in his popular "Mathematical Games" column in the October 1973 issue of the *Scientific American,* discussed PWWs as "look-see" diagrams. Gardner points out that "in many cases a dull proof can be supplemented by a geometric analogue so simple and beautiful that the truth of a theorem is almost seen at a glance." This dramatically illustrates the dictionary quote above: in English "to see" is often "to understand."

In the same vein, the editorial policy of *The College Mathematics Journal* throughout most of the 1980s stated that, in addition to expository articles, "The Journal also invites other types of contributions, most notably: *proofs without words,* mathematical poetry, quotes, ..." (their italics). But PWWs are not recent innovations — they have a long history. Indeed, in this volume you will find modern renditions of proofs without words from ancient China, classical Greece, and India of the twelfth century.

Of course, "proofs without words" are not really proofs. As Theodore Eisenberg and Tommy Dreyfus note in their paper "On the Reluctance to Visualize in Mathematics" [in *Visualization in Teaching and Learning Mathematics*, MAA Notes Number 19], some consider such visual arguments to be of little value, and "that there is one and only one way to communicate mathematics, and 'proofs without words' are not acceptable." But to counter this viewpoint, Eisenberg and Dreyfus go on to give us some quotes on the subject:

> [Paul] Halmos, speaking of Solomon Lefschetz (editor of the *Annals*), stated: "He saw mathematics not as logic but as pictures." Speaking of what it takes to be a mathematician, he stated: "To be a scholar of mathematics you must be born with ... the ability to visualize" and most teachers try to develop this ability in their students. [George] Pólya's "Draw a figure ..." is classic pedagogic advice, and Einstein and Poincaré's views that we should use our visual intuitions are well known.

So, if "proofs without words" are not proofs, what are they? As you will see from this collection, this question does not have a simple, concise answer. But generally, PWWs are pictures or diagrams that help the observer see *why* a particular statement may be true, and also to see *how* one might begin to go about proving it true. In some an equation or two may appear in order to guide the observer in this process. But the emphasis is clearly on providing visual clues to the observer to stimulate mathematical thought.

I should note that this collection is not intended to be complete. It does not include all PWWs which have appeared in print, but is rather a sample representative of the genre. In addition, as readers of the Association's journals are well aware, new PWWs appear in print rather frequently, and I anticipate that this will continue. Perhaps some day a second volume of PWWs will appear!

I hope that the readers of this collection will find enjoyment in discovering or rediscovering some elegant visual demonstrations of certain mathematical ideas; that teachers will want to share many of them with their students; and that all will find stimulation and encouragement to try to create new "proofs without words."

Acknowledgment. I would like to express my appreciation and gratitude to the many people who have played a part in the publication of this collection: to Gerald Alexanderson and Martha Siegel, who, as editors of *Mathematics Magazine*, gave me encouragement over the years as I learned to read and write PWWs; to Doris Schattschneider, Eugene Klotz, and Richard Guy for sharing with me their collections of PWWs; and finally, to all those individuals who have contributed "proofs without words" to the mathematical literature (see the *Index of Names* on pp. 151-152), without whom this collection simply would not exist.

Note. All the drawings in this collection were redone to create a uniform appearance. In a few instances titles were changed, and shading or symbols were added (or deleted) for clarity. Any errors resulting from that process are entirely my responsibility.

Roger B. Nelsen
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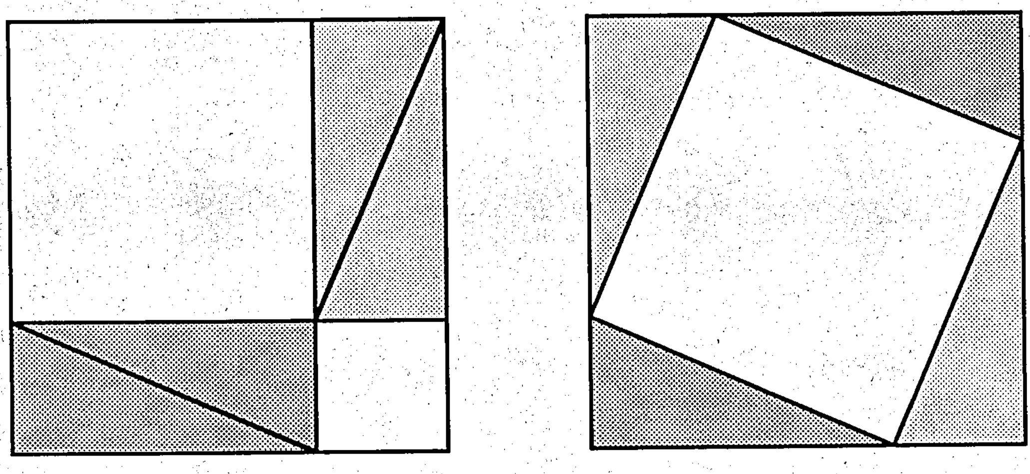
Portland, Oregon

Contents

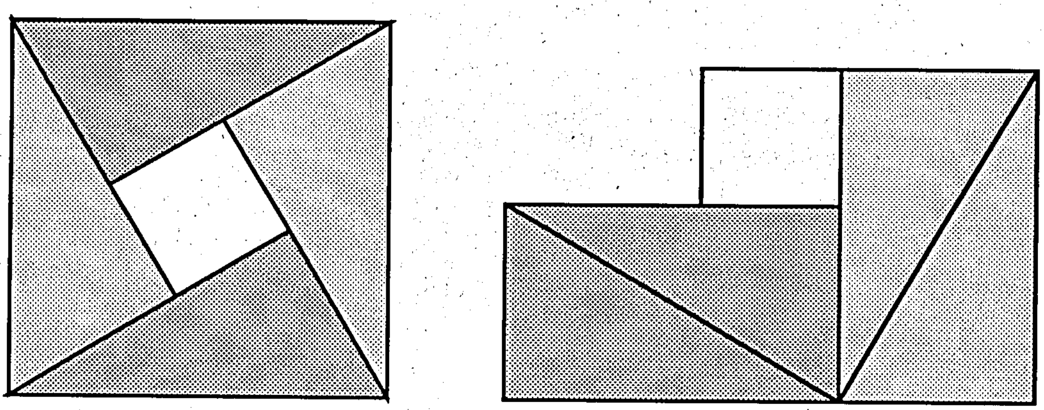
Geometry & Algebra

The Pythagorean Theorem I



—adapted from the *Chou pei suan ching*
(author unknown, circa B.C. 200?)

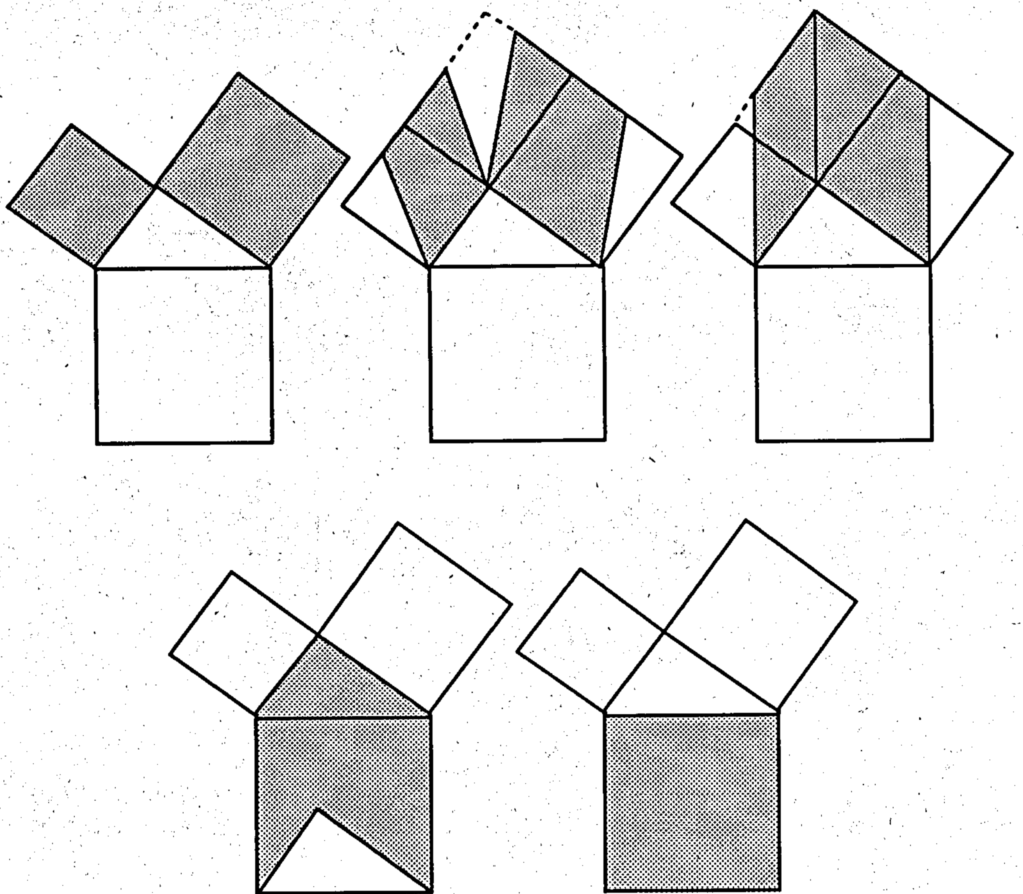
The Pythagorean Theorem II



Behold!

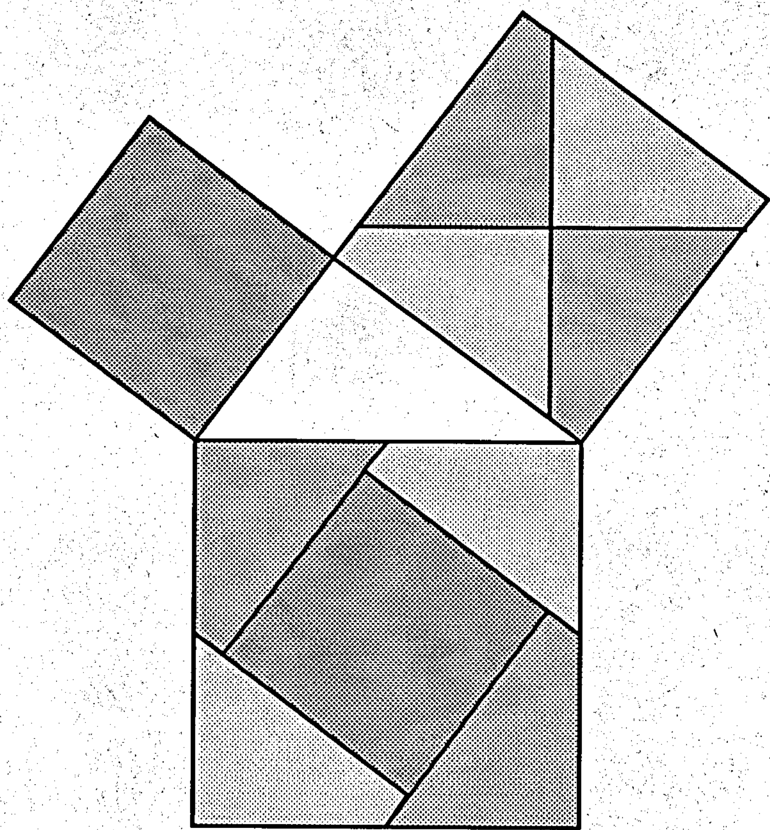
—Bhāskara (12th century)

The Pythagorean Theorem III



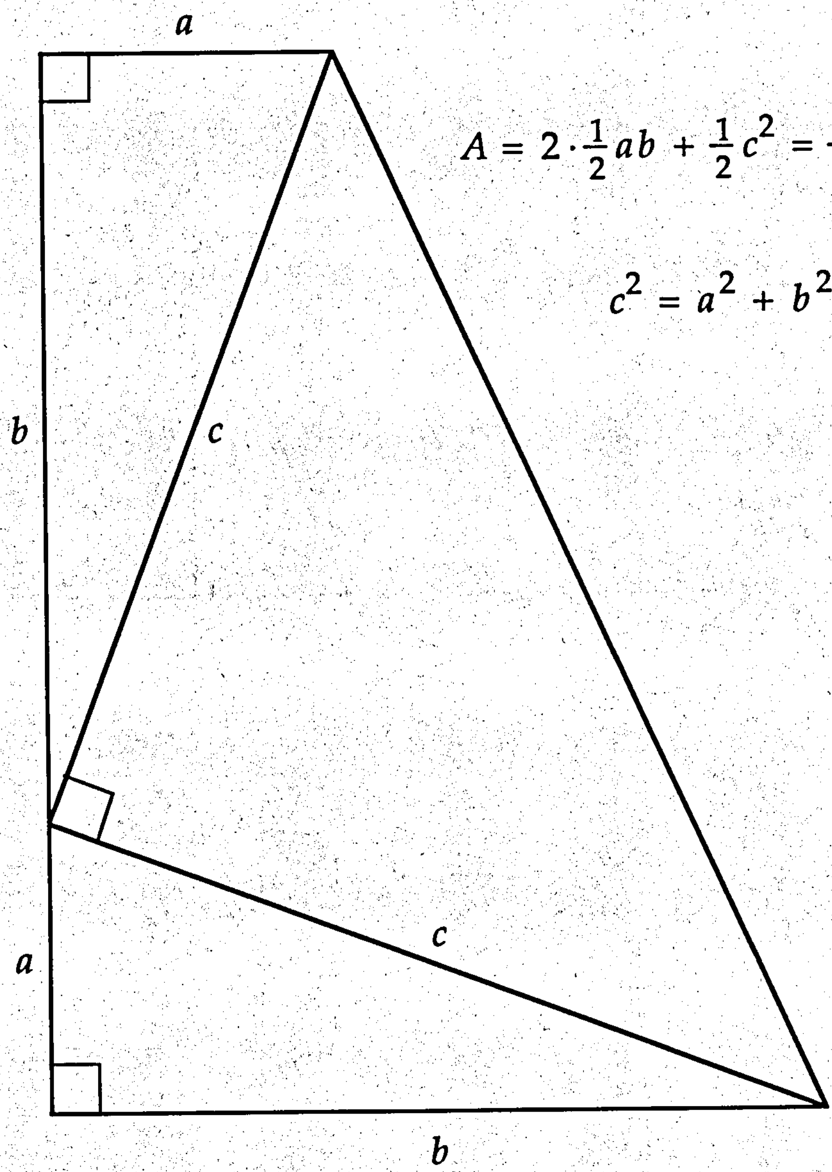
—based on Euclid's proof

The Pythagorean Theorem IV



—H. E. Dudeney (1917)

The Pythagorean Theorem V



$$A = 2 \cdot \frac{1}{2} ab + \frac{1}{2} c^2 = \frac{1}{2}(a + b)^2$$

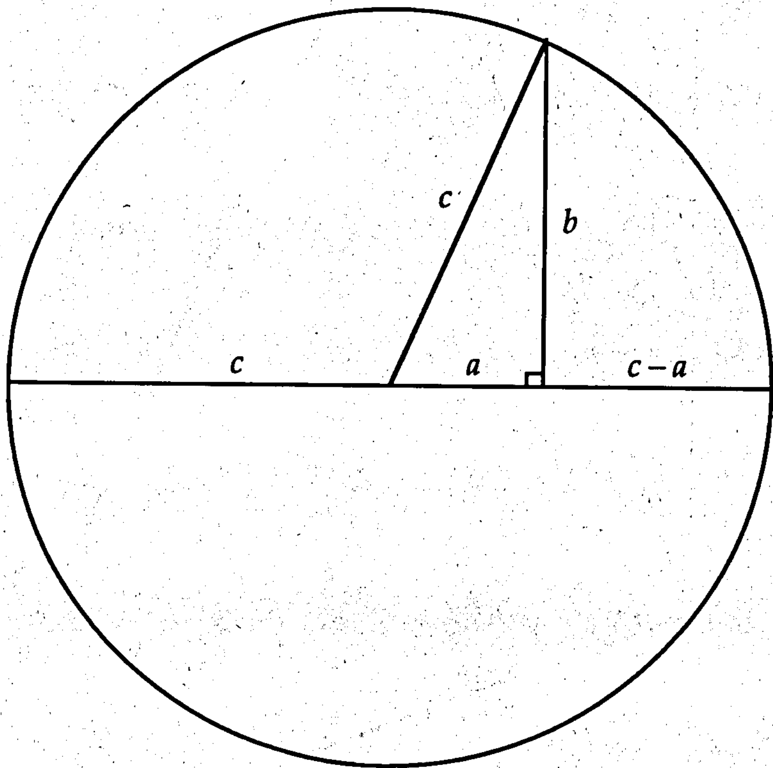
$$c^2 = a^2 + b^2$$

—James A. Garfield (1876)
20th President of the United States

The Pythagorean Theorem VI

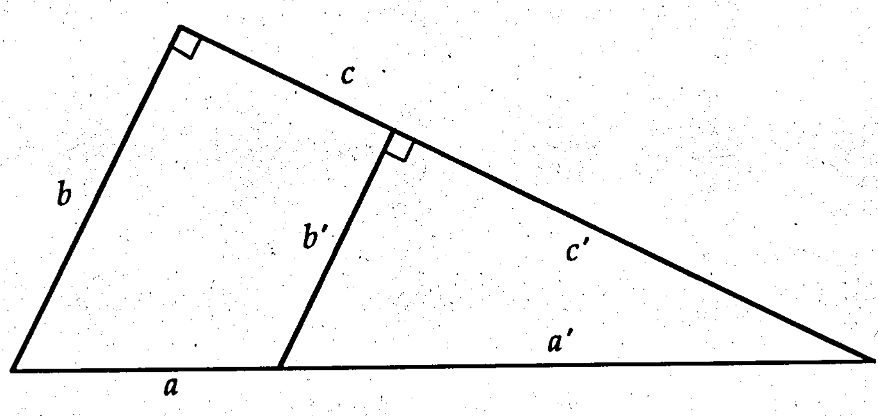
$$\frac{c+a}{b} = \frac{b}{c-a}$$

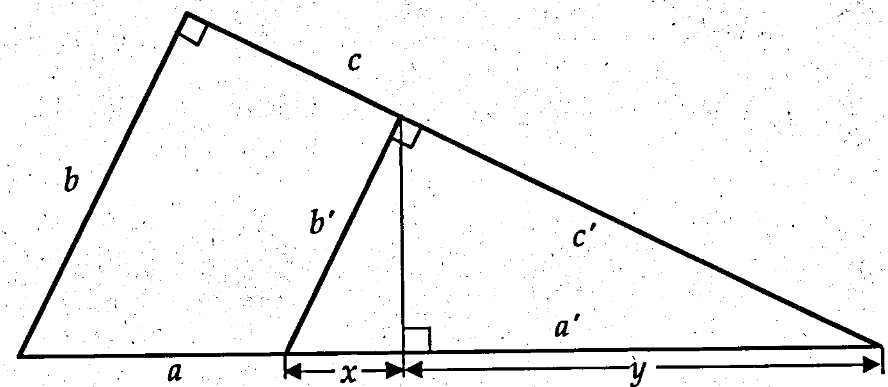
$$a^2 + b^2 = c^2$$



—Michael Hardy

A Pythagorean Theorem: $a \cdot a' = b \cdot b' + c \cdot c'$





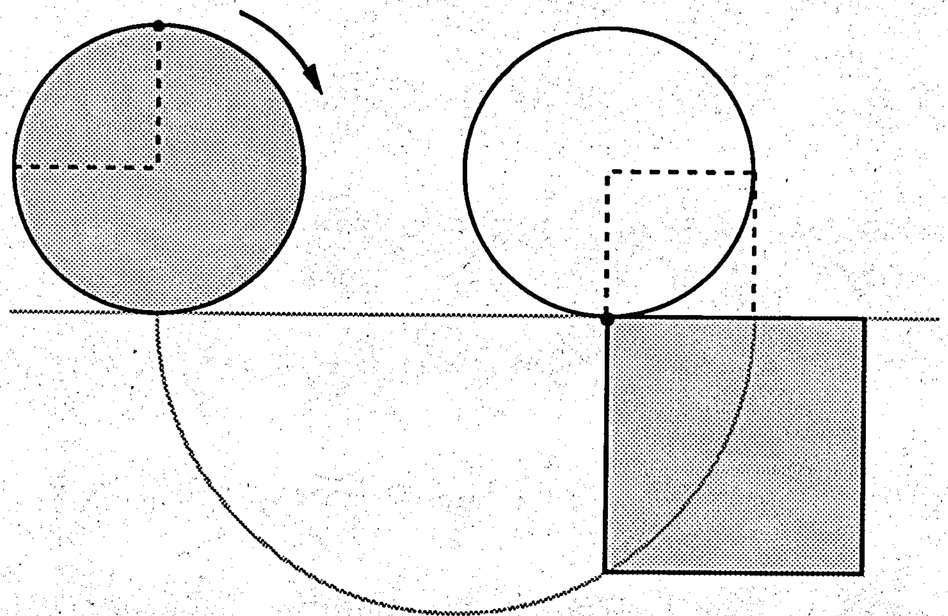
$$\frac{x}{b} = \frac{b'}{a} \Rightarrow a \cdot x = b \cdot b';$$

$$\frac{y}{c} = \frac{c'}{a} \Rightarrow a \cdot y = c \cdot c';$$

$$\therefore\ a \cdot a' = a \cdot (x + y) = b \cdot b' + c \cdot c'.$$

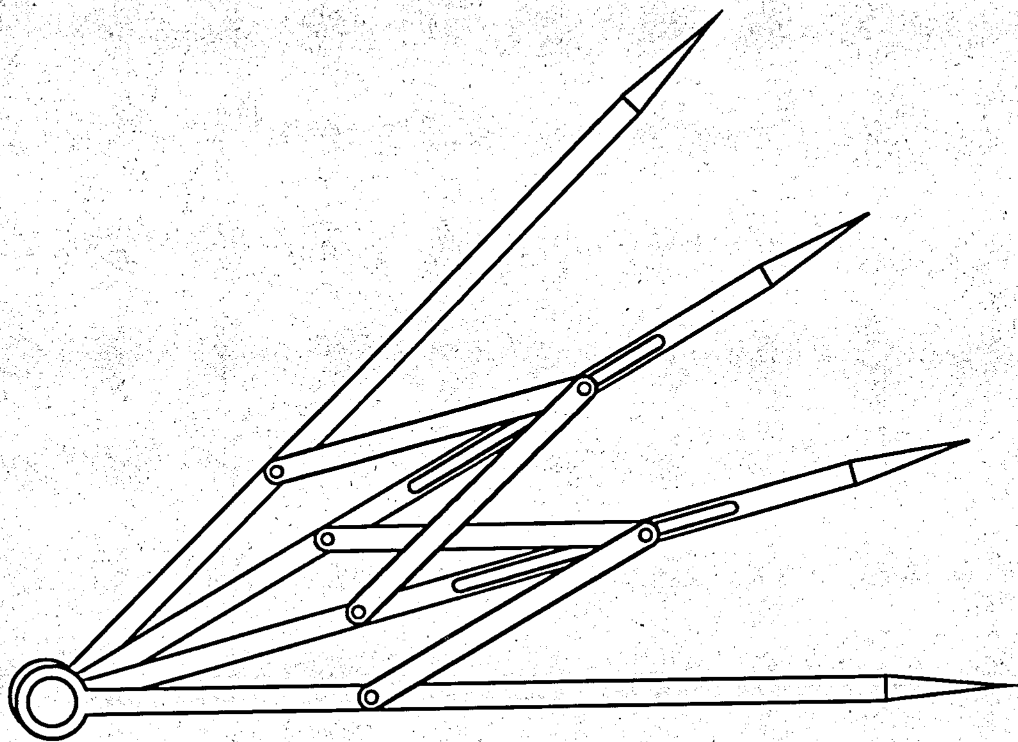
—Enzo R. Gentile

The Rolling Circle Squares Itself



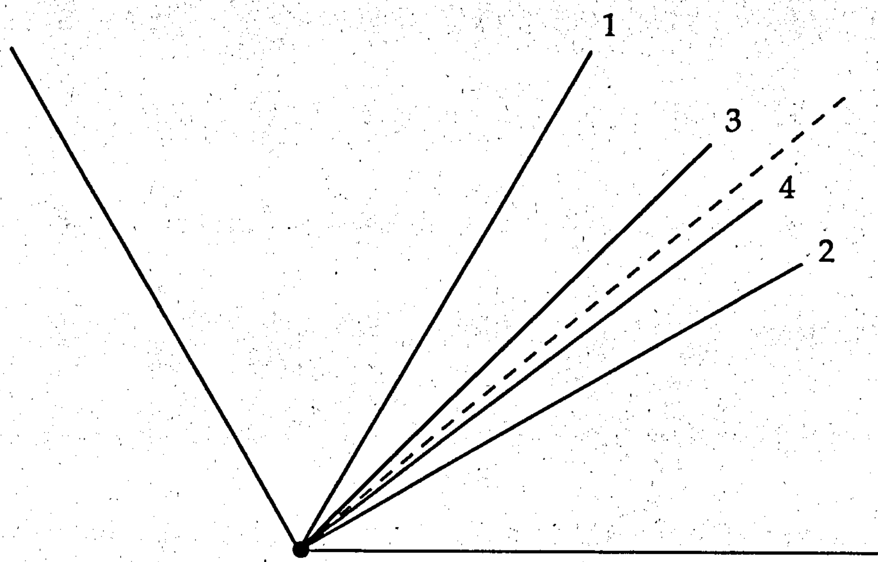
—Thomas Elsner

On Trisecting an Angle



—Rufus Isaacs

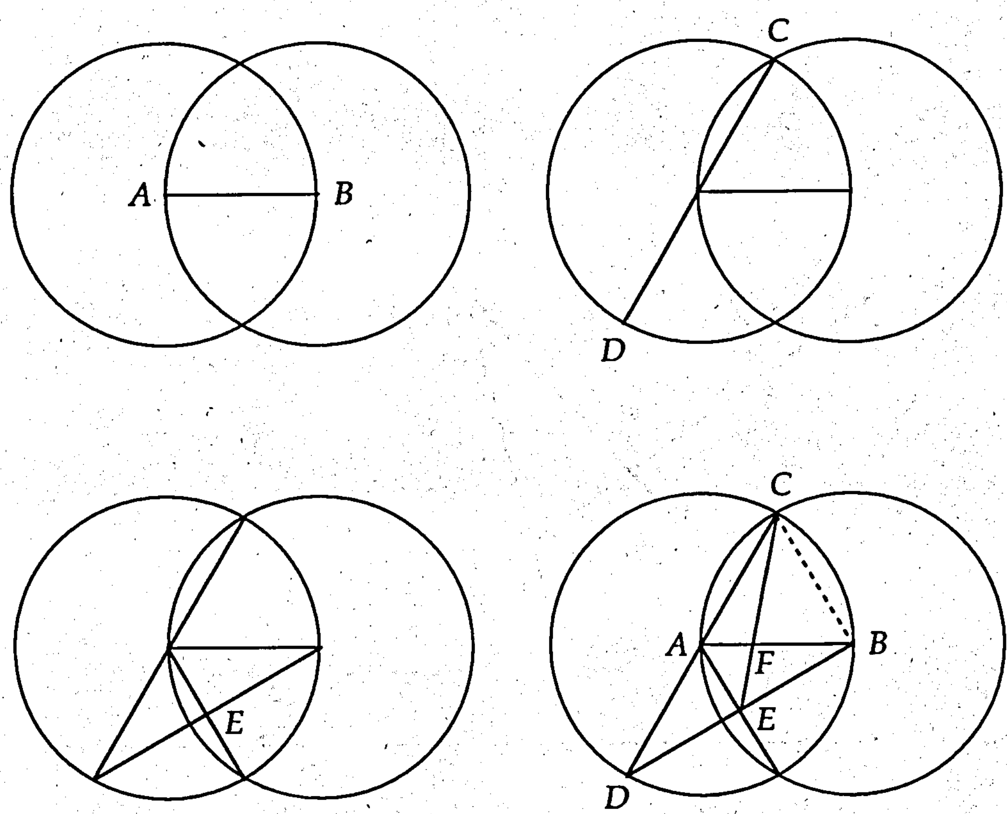
Trisection of an Angle in an Infinite Number of Steps



$$\frac{1}{3} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots$$

—Eric Kincanon

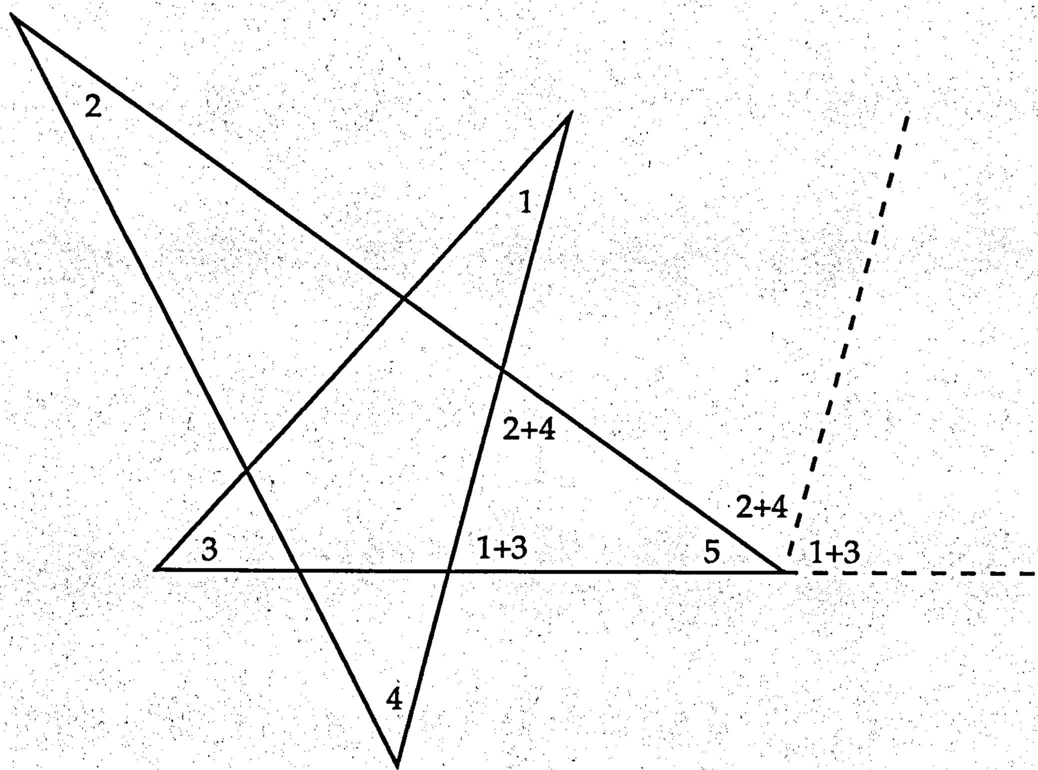
Trisection of a Line Segment



$$\overline{AF} = \frac{1}{3} \cdot \overline{AB}$$

—Scott Coble

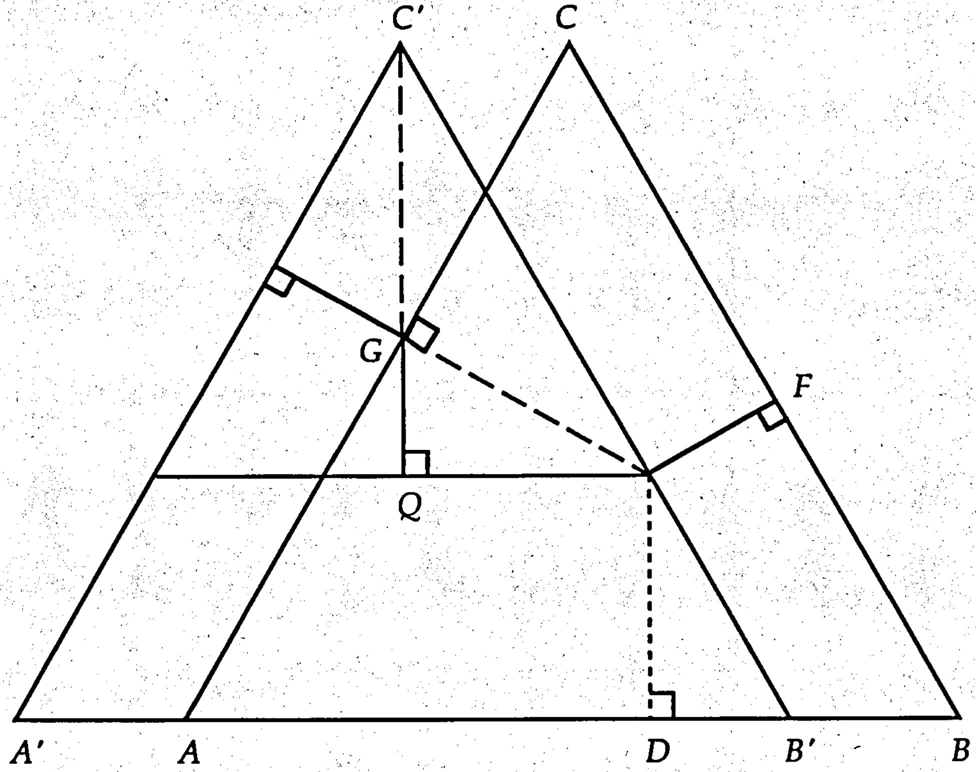
The Vertex Angles of a Star Sum to 180°



—Fouad Nakhli

Viviani's Theorem

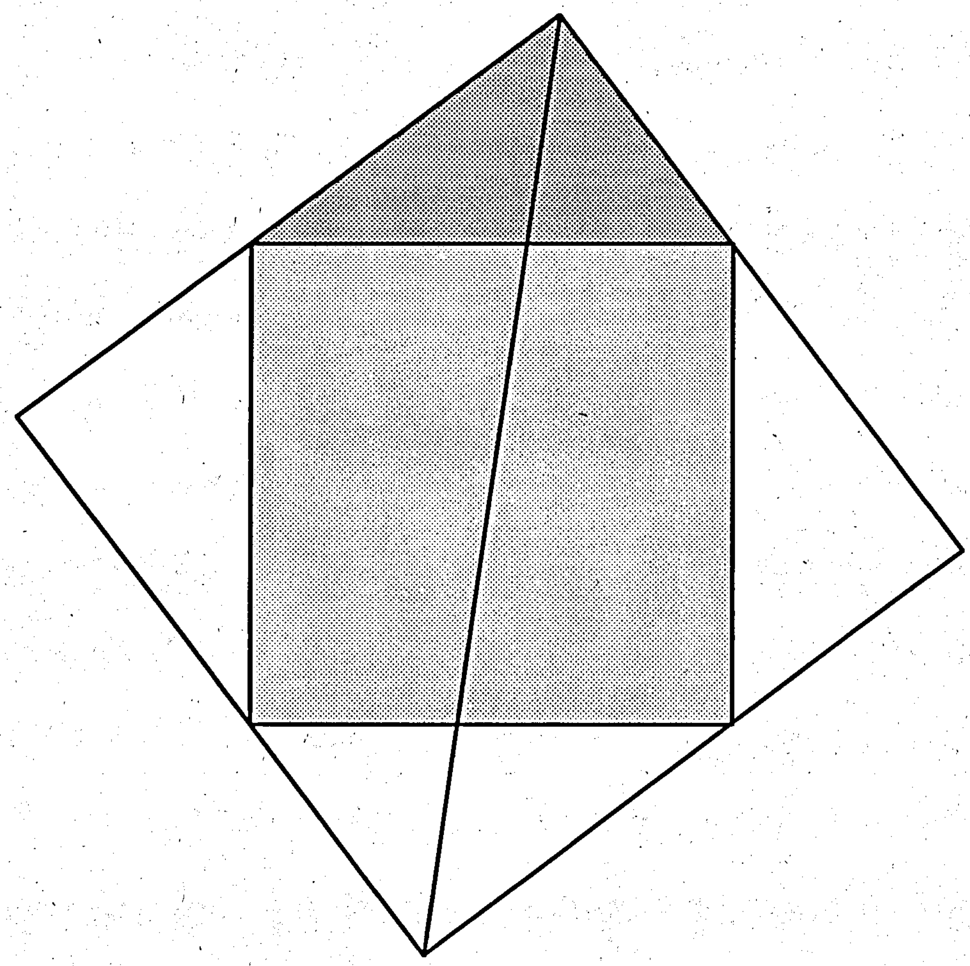
The perpendiculars to the sides from a point on the boundary or within an equilateral triangle add up to the height of the triangle.



—Samuel Wolf

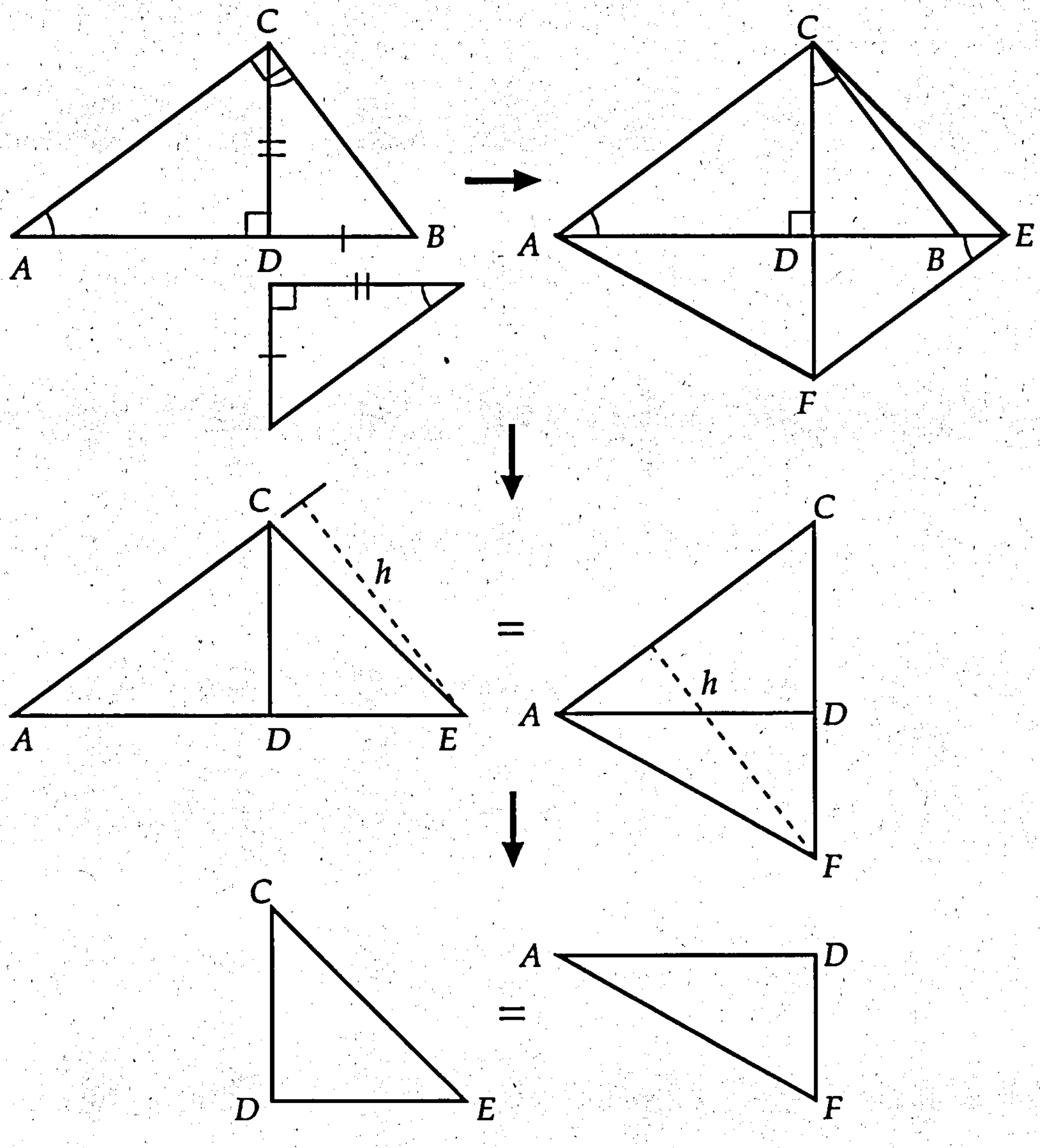
A Theorem About Right Triangles

The internal bisector of the right angle of a right triangle bisects the square on the hypotenuse.



—Roland H. Eddy

Area and the Projection Theorem of a Right Triangle

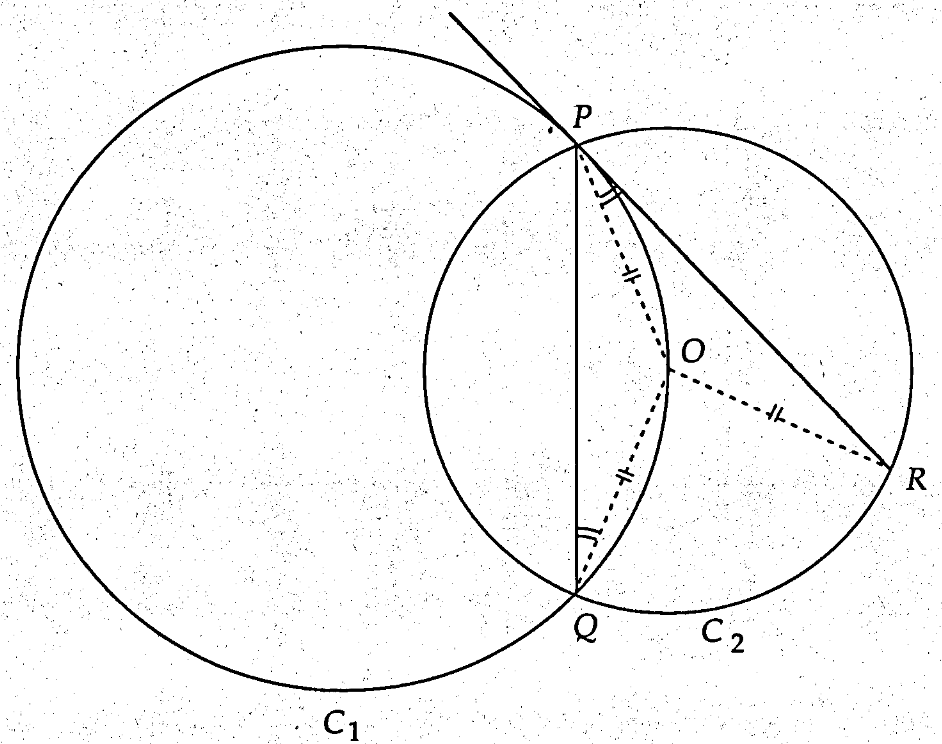


$$CD^2 = AD \cdot DB$$

—Sidney H. Kung

Chords and Tangents of Equal Length

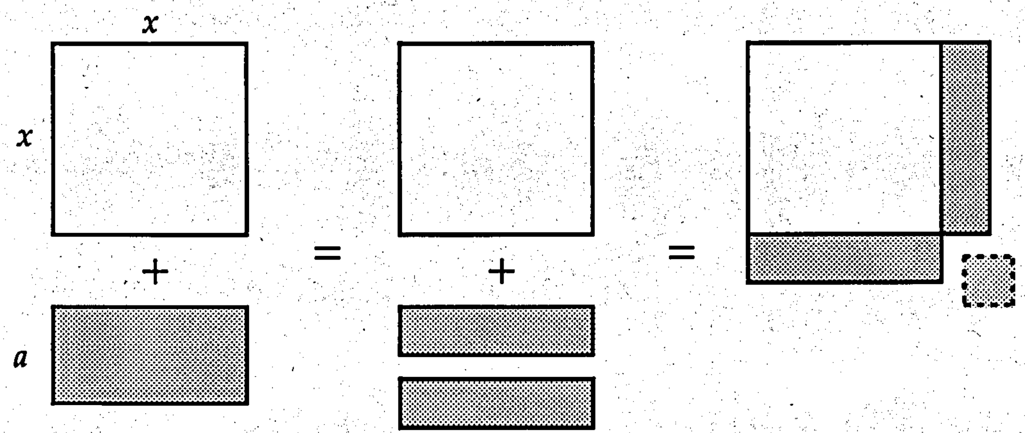
If circle C_1 passes through the center O of circle C_2, the length of the common chord $\overline{PQ}$ is equal to the tangent segment $\overline{PR}$.



—Roland H. Eddy

Completing the Square

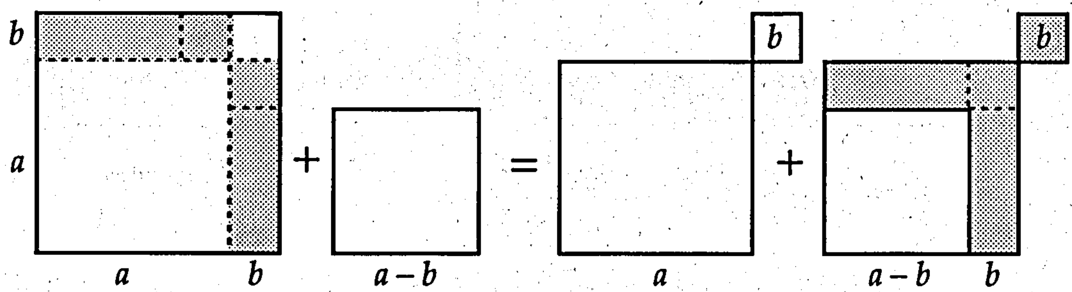
$$x^2 + ax = (x + a/2)^2 - (a/2)^2$$



—Charles D. Gallant

Algebraic Areas I

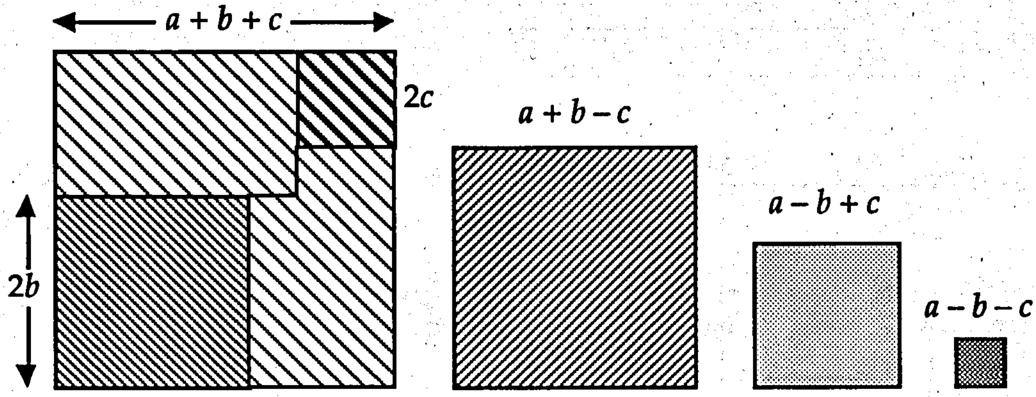
$$(a+b)^2 + (a-b)^2 = 2(a^2+b^2)$$

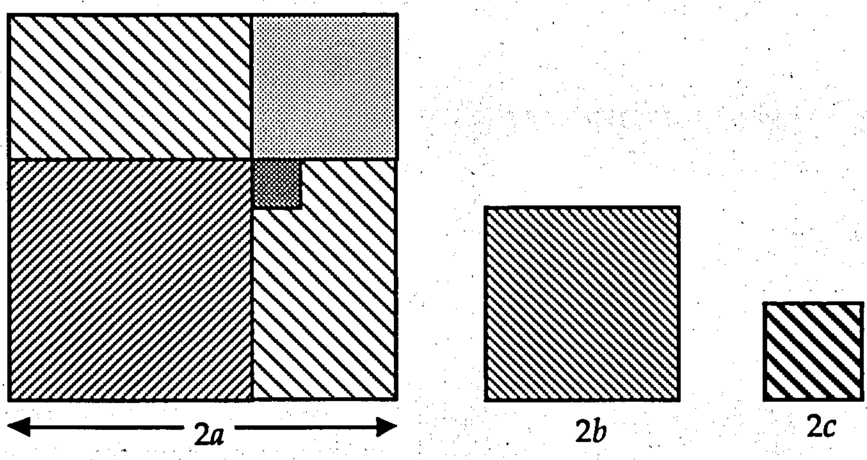


—Shirley Wakin

Algebraic Areas II

$$(a+b+c)^2 + (a+b-c)^2 + (a-b+c)^2 + (a-b-c)^2$$
$$= (2a)^2 + (2b)^2 + (2c)^2$$

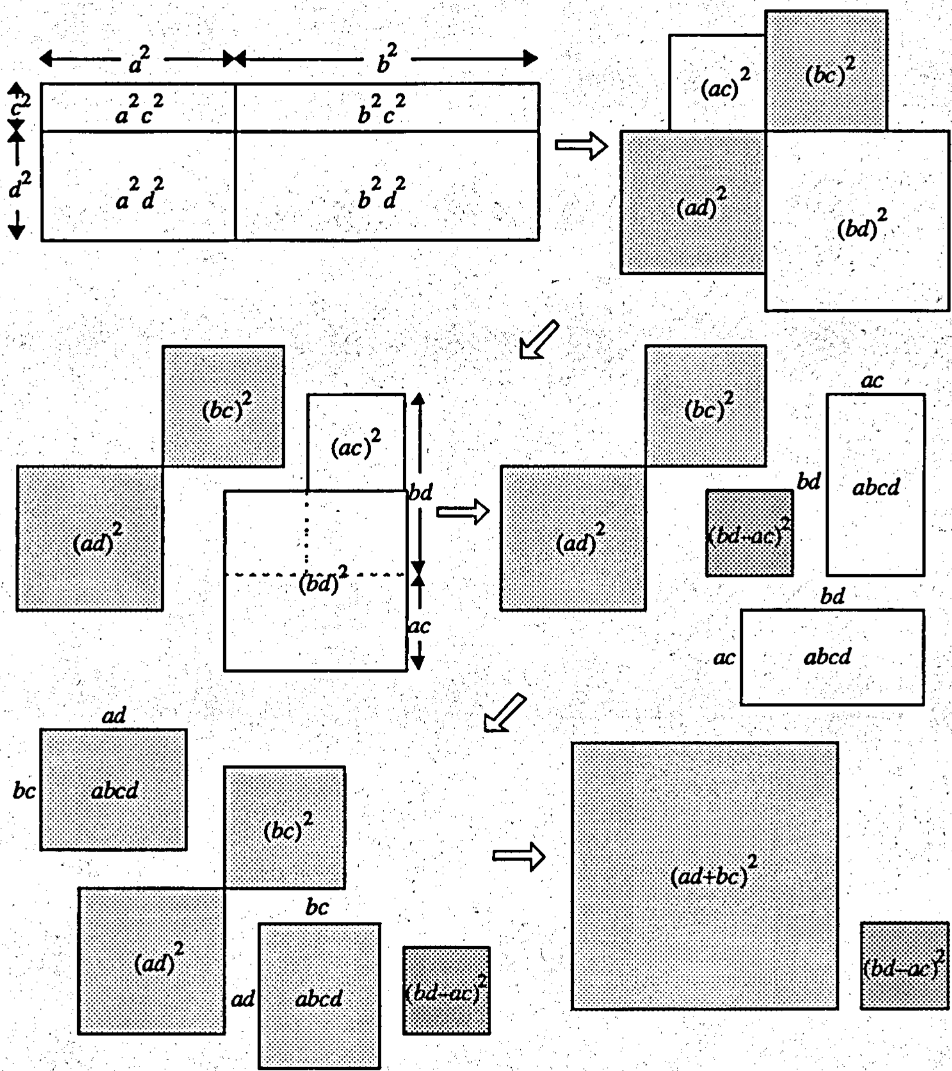




—Sam Pooley and K. Ann Drude

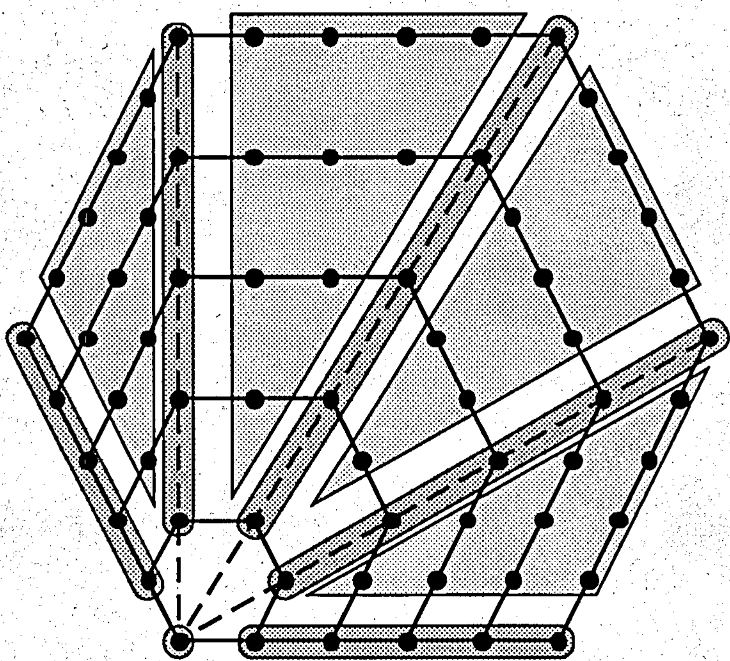
Diophantus of Alexandria's "Sum of Squares" Identity

$$(a^2 + b^2)(c^2 + d^2) = (ad + bc)^2 + (bd - ac)^2$$



—RBN

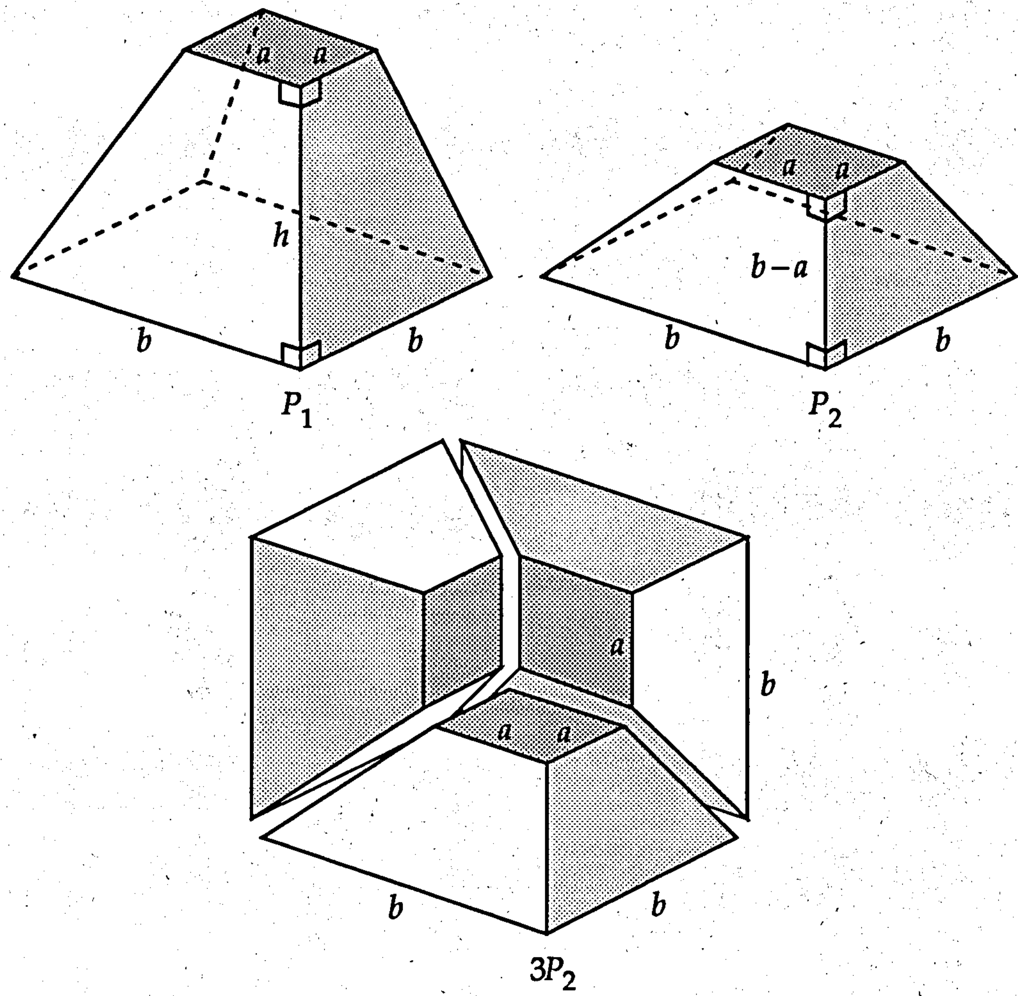
The k^{th} n-gonal Number is $1 + (k-1)(n-1) + \frac{1}{2}(k-2)(k-1)(n-2)$



—Dave Logothetti

The Volume of a Frustum of a Square Pyramid

[Problem 14, *The Moscow Papyrus,* circa 1850 B.C.]



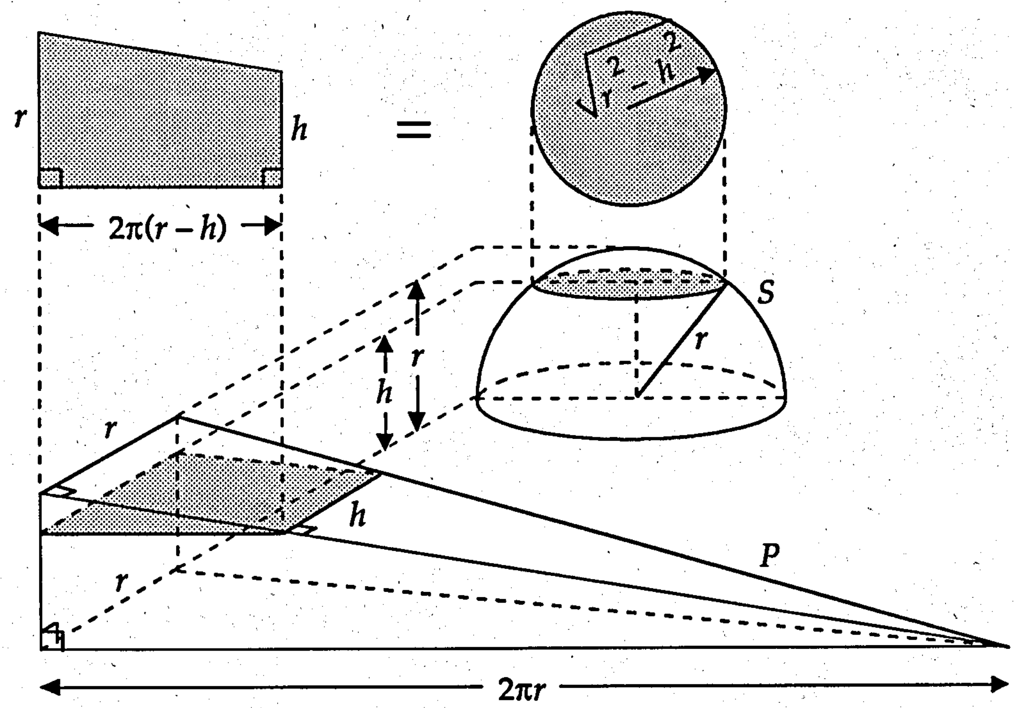
$$V(P_1) = \frac{h}{b-a} V(P_2) = \frac{h}{b-a} \cdot \frac{1}{3}(b^3 - a^3) = \frac{h}{3}(a^2 + ab + b^2)$$

REFERENCES

1. C. B. Boyer, *A History of Mathematics,* John Wiley & Sons, New York, 1968, pp. 20-22.
2. R. J. Gillings, *Mathematics in the Time of the Pharaohs,* The MIT Press, Cambridge, 1972, pp. 187-193.

—RBN

The Volume of a Hemisphere via Cavalieri's Principle*



$$V_S = V_P = \frac{1}{3}r^2 \cdot 2\pi r = \frac{2}{3}\pi r^3$$

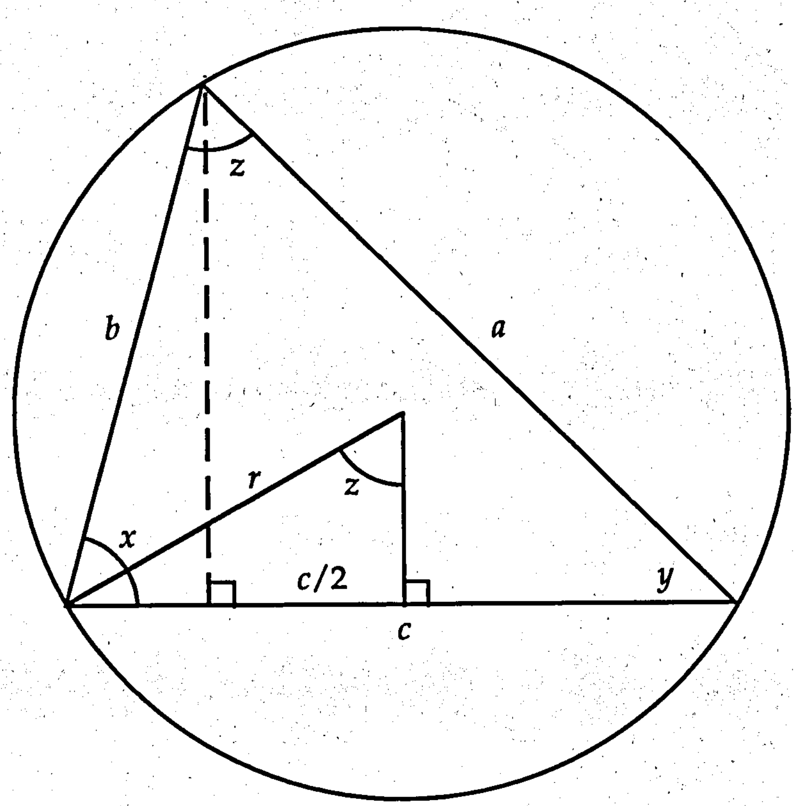
*Tzu Geng, son of the most celebrated mathematician Tzu Chung Chih in ancient China, was believed to be the first to develop the principle in the 5th century A. D.

—Sidney H. Kung

Trigonometry, Calculus & Analytic Geometry

Sine of the Sum

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \quad \text{for } x+y<\pi$$



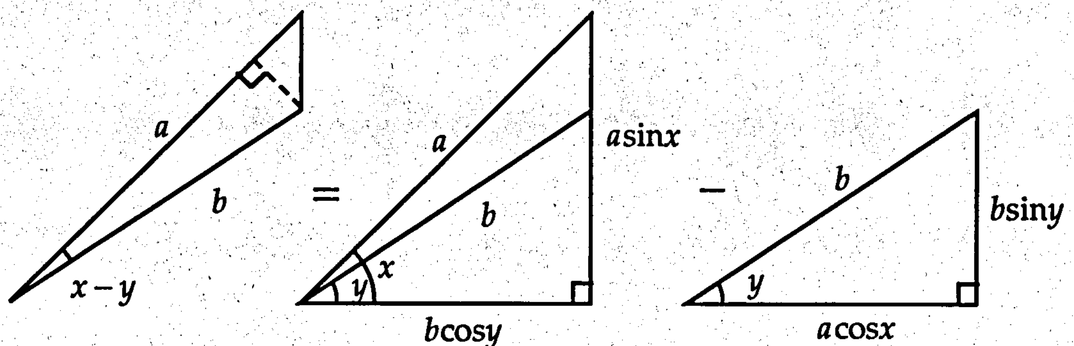
$$c = a\cos y + b\cos x$$

$$r = 1/2 \Rightarrow \sin z = (c/2)/(1/2) = c,\ \sin x = a,\ \sin y = b;$$

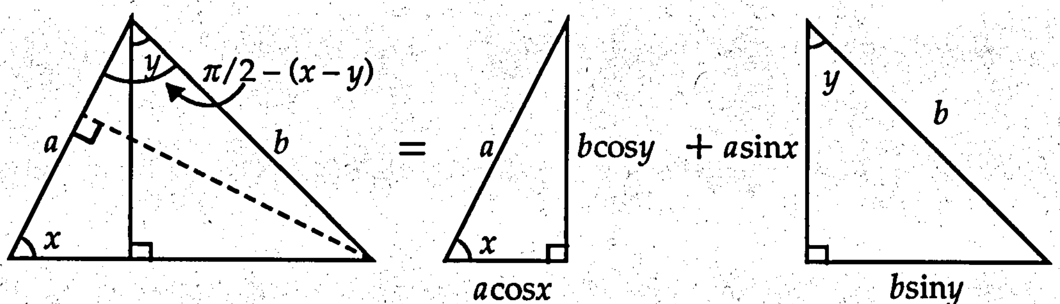
$$\sin(x+y) = \sin(\pi-(x+y)) = \sin z = \sin x\cos y + \sin y\cos x$$

—Sidney H. Kung

Area and Difference Formulas



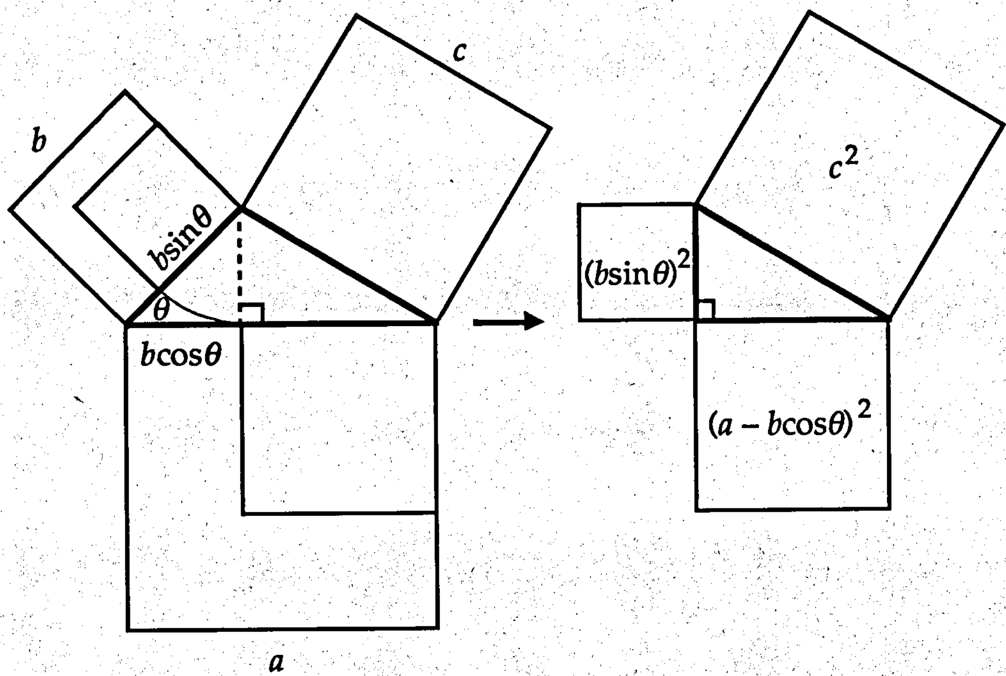
$$\sin(x-y) = \sin x\cos y - \cos x\sin y$$



$$\cos(x-y) = \cos x\cos y + \sin x\sin y$$

—Sidney H. Kung

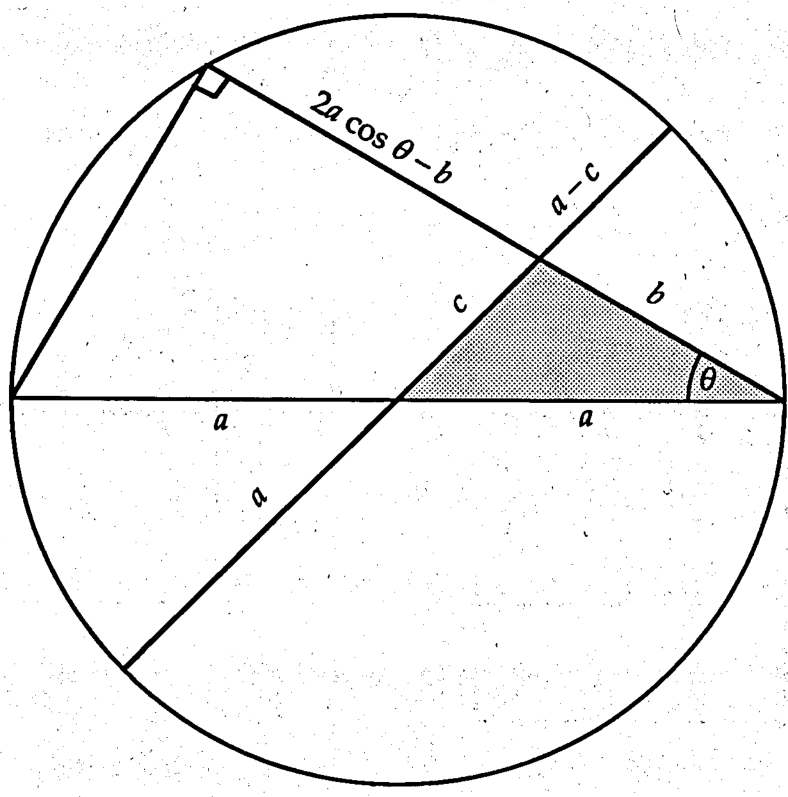
The Law of Cosines I



$$\begin{aligned} c^2 &= (b \sin \theta)^2 + (a - b \cos \theta)^2 \\ &= a^2 + b^2 - 2ab \cos \theta \end{aligned}$$

—Timothy A. Sipka

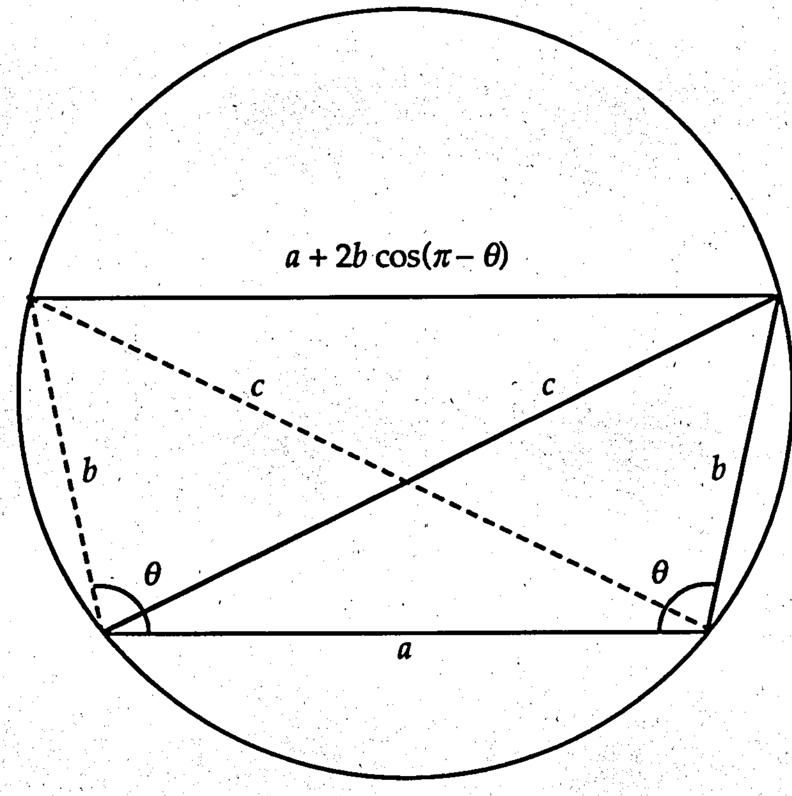
The Law of Cosines II



$$(2a\cos\theta - b)b = (a-c)(a+c)$$
$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

—Sidney H. Kung

The Law Of Cosines III (via Ptolemy's Theorem)

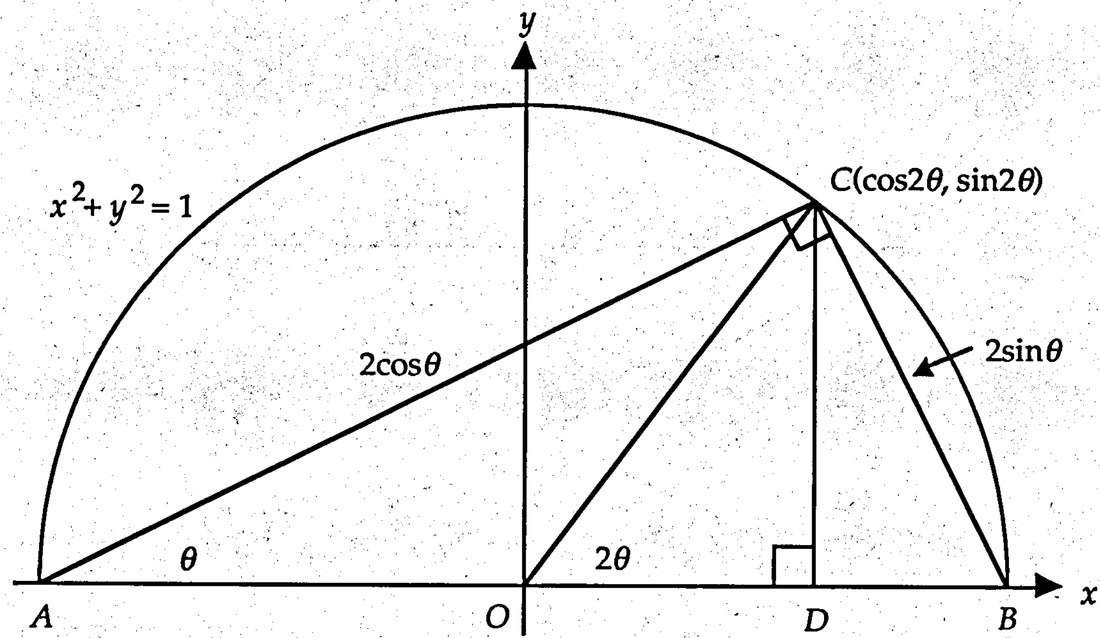


$$c \cdot c = b \cdot b + (a + 2b\cos(\pi - \theta)) \cdot a$$

$$c^2 = a^2 + b^2 - 2ab \cdot \cos\theta$$

—Sidney H. Kung

The Double-Angle Formulas



$$\Delta ACD \sim \Delta ABC$$

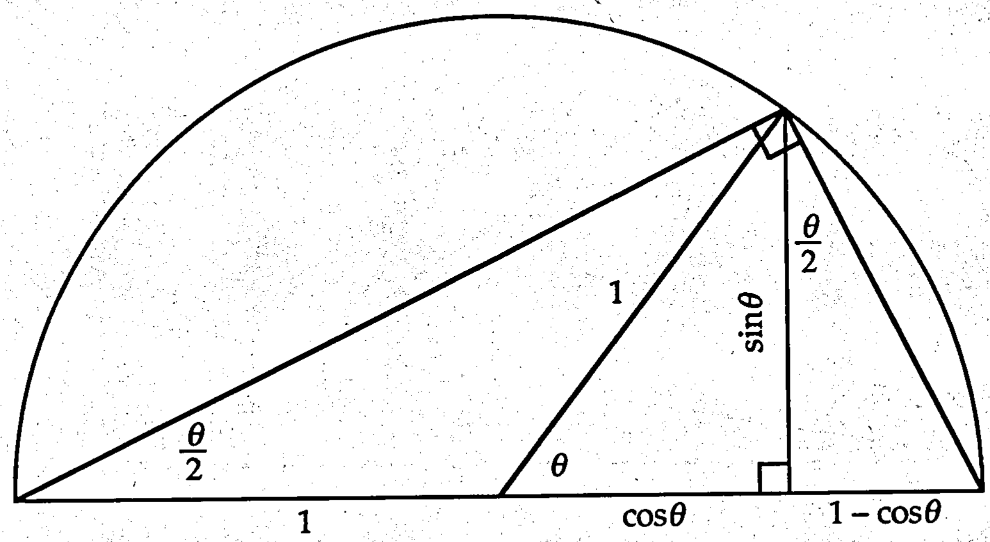
$$\overline{CD} / \overline{AC} = \overline{BC} / \overline{AB} \qquad \overline{AD} / \overline{AC} = \overline{AC} / \overline{AB}$$

$$\sin 2\theta / 2\cos\theta = 2\sin\theta/2 \qquad (1 + \cos 2\theta)/2\cos\theta = 2\cos\theta/2$$

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos 2\theta = 2\cos^2\theta - 1$$

—RBN

The Half-Angle Tangent Formulas

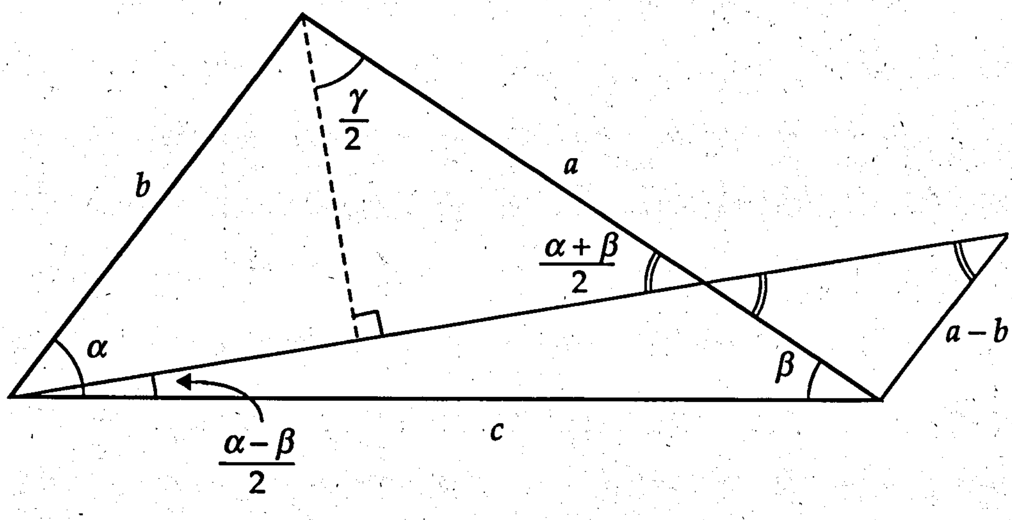


$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$$

—R. J. Walker

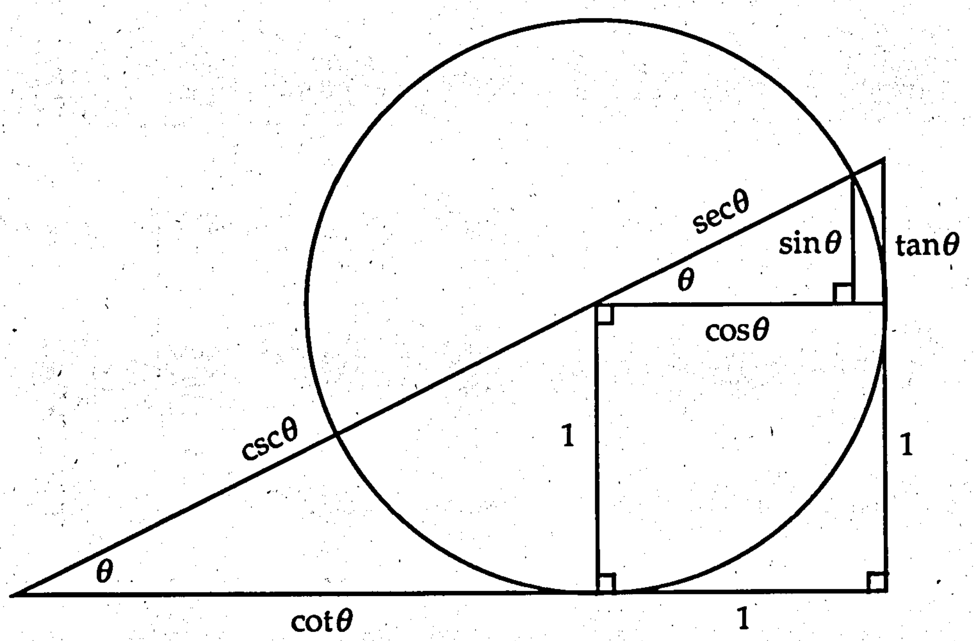
Mollweide's Equation

$$(a-b)\cos\frac{\gamma}{2} = c\sin\left(\frac{\alpha-\beta}{2}\right)$$



—H. Arthur DeKleine

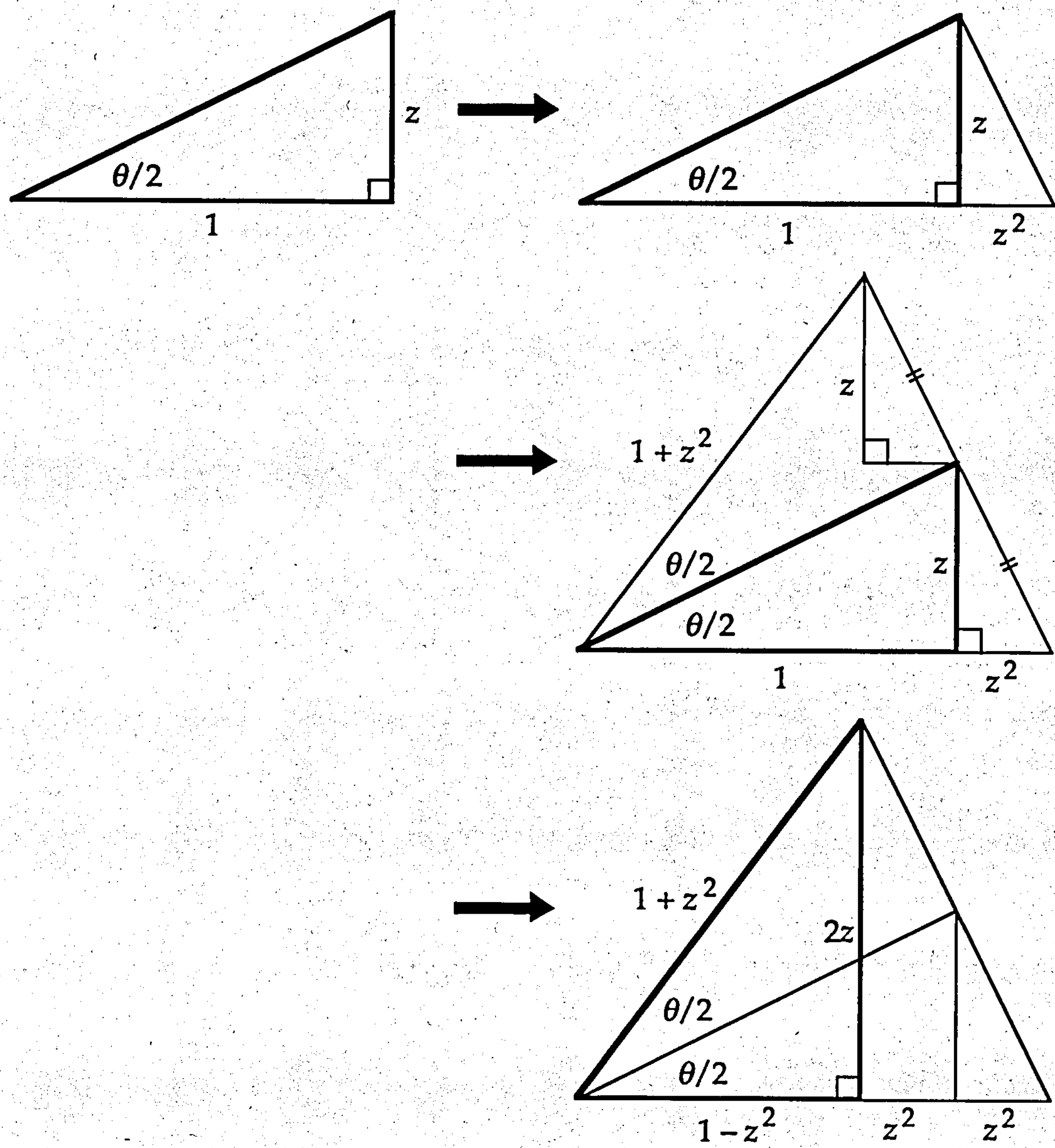
$$(\tan\theta + 1)^2 + (\cot\theta + 1)^2 = (\sec\theta + \csc\theta)^2$$



$$\tan^2\theta + 1 = \sec^2\theta$$
$$\cot^2\theta + 1 = \csc^2\theta$$
$$(\tan\theta + 1)^2 + (\cot\theta + 1)^2 = (\sec\theta + \csc\theta)^2$$
$$\left(\text{also } \tan\theta = \frac{\tan\theta + 1}{\cot\theta + 1}\right)$$

—William Romaine

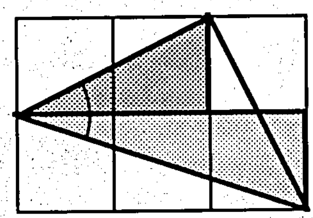
The Substitution to Make a Rational Function of the Sine and Cosine



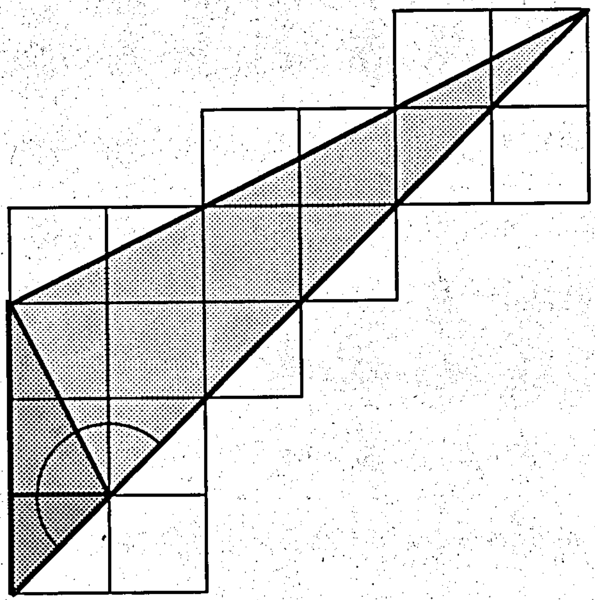
$$z = \tan\frac{\theta}{2} \Rightarrow \sin\theta = \frac{2z}{1+z^2} \text{ and } \cos\theta = \frac{1-z^2}{1+z^2}$$

—RBN

Sums of Arctangents



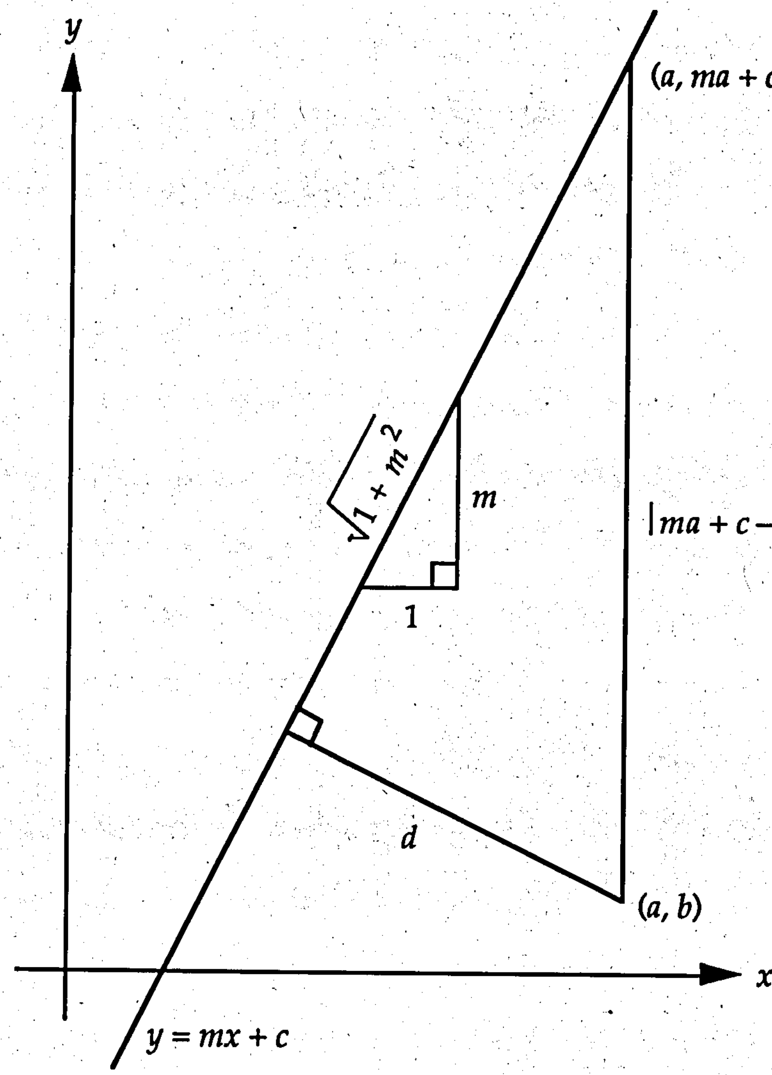
$$\arctan\frac{1}{2} + \arctan\frac{1}{3} = \frac{\pi}{4}$$



$$\arctan 1 + \arctan 2 + \arctan 3 = \pi$$

—Edward M. Harris

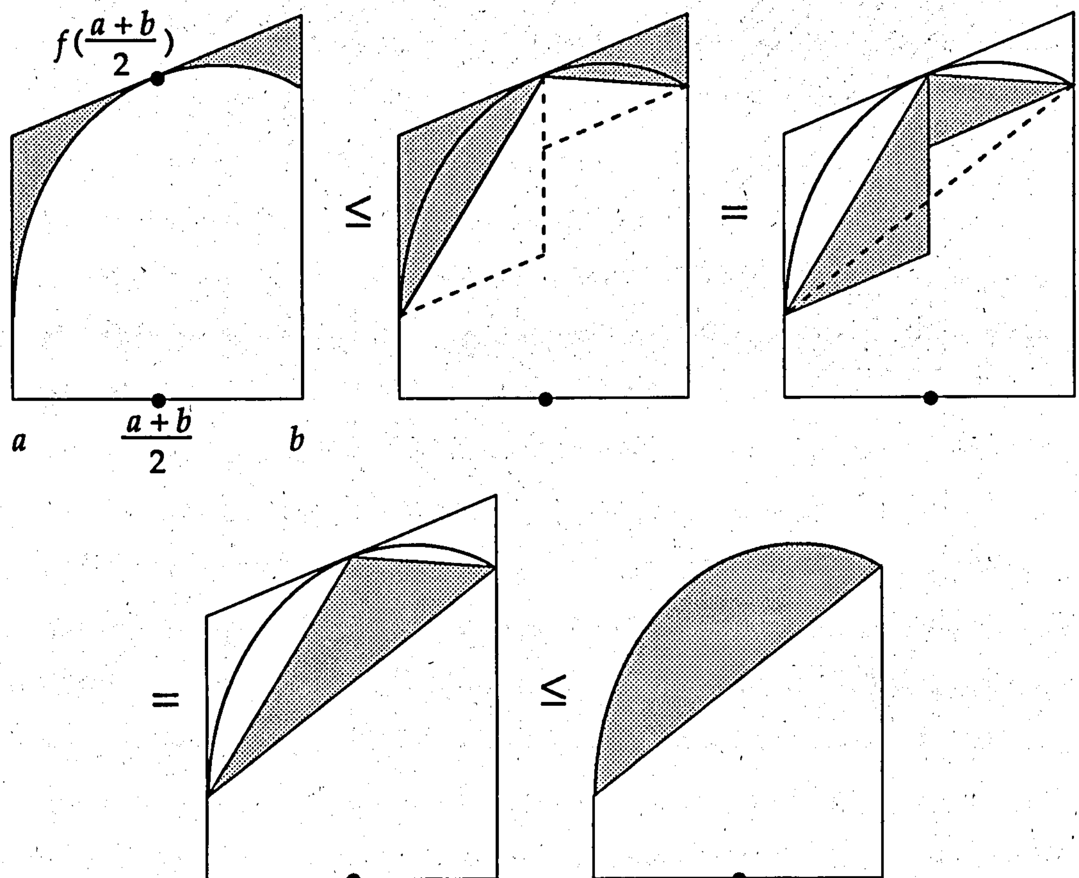
The Distance Between a Point and a Line



$$\frac{d}{1} = \frac{|ma + c - b|}{\sqrt{1 + m^2}}$$

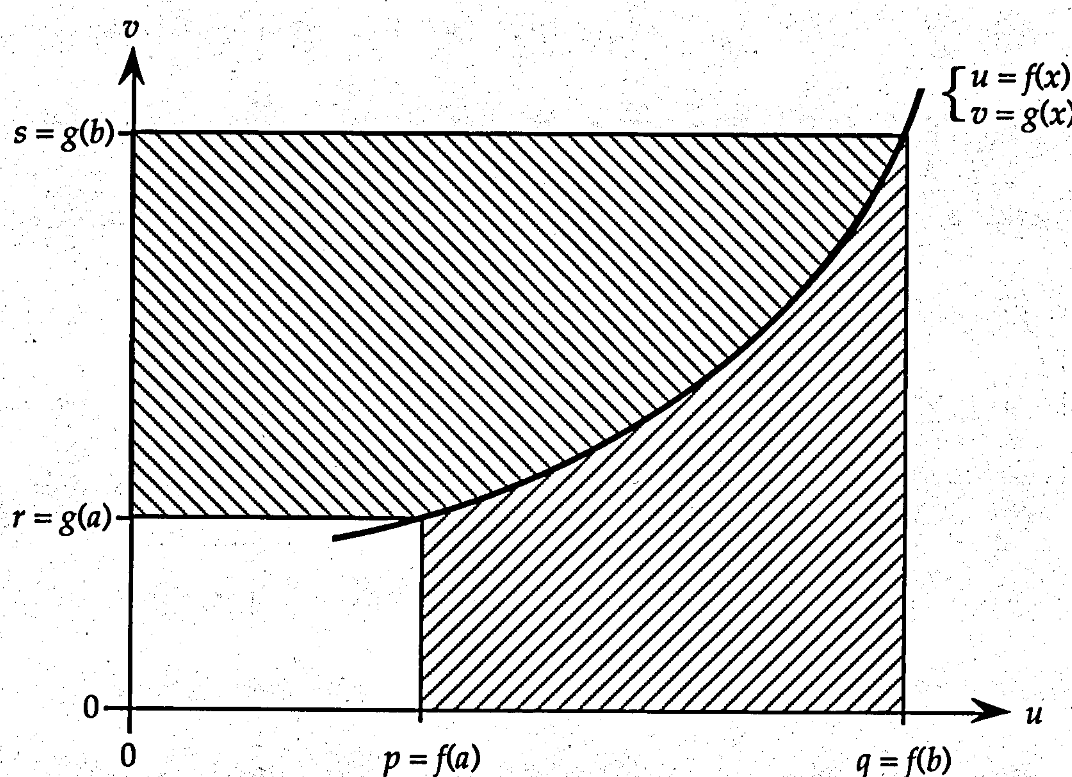
—R. L. Eisenman

The Midpoint Rule is Better than the Trapezoidal Rule for Concave Functions



—Frank Burk

Integration by Parts



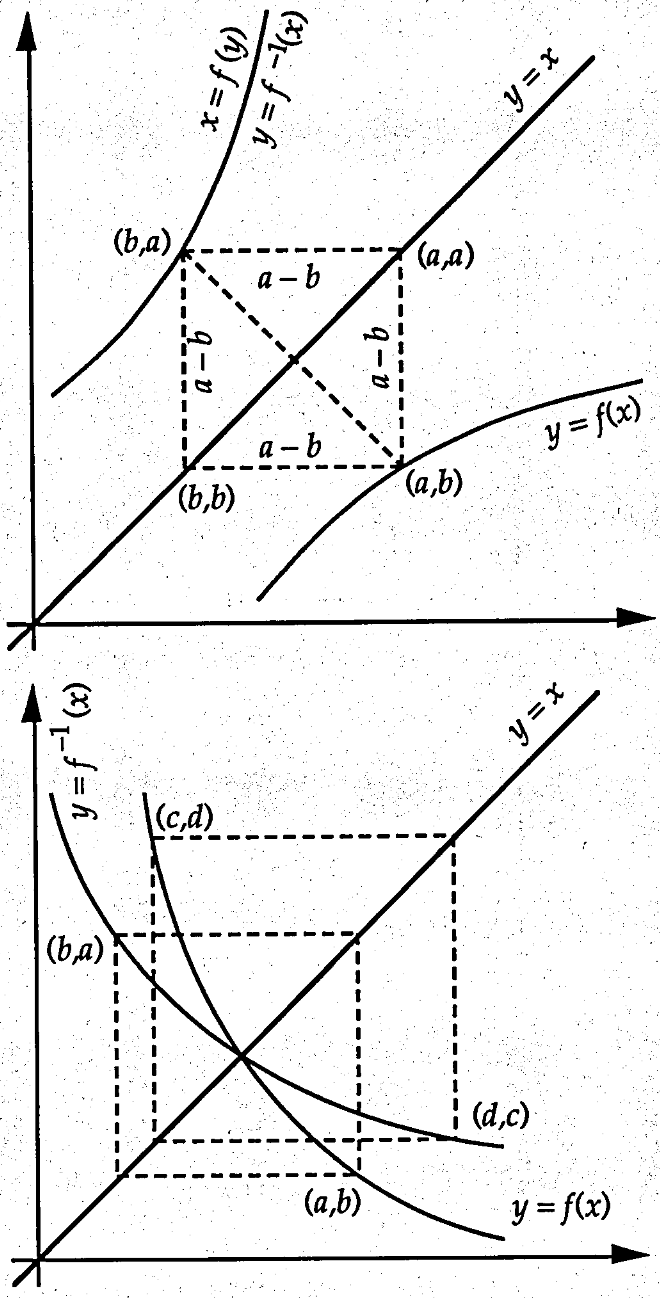
Area [\\\] + Area [///] = $qs - pr$

$$\int_r^s u\,dv + \int_p^q v\,du = uv\Big|_{(p,r)}^{(q,s)}$$

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)dx$$

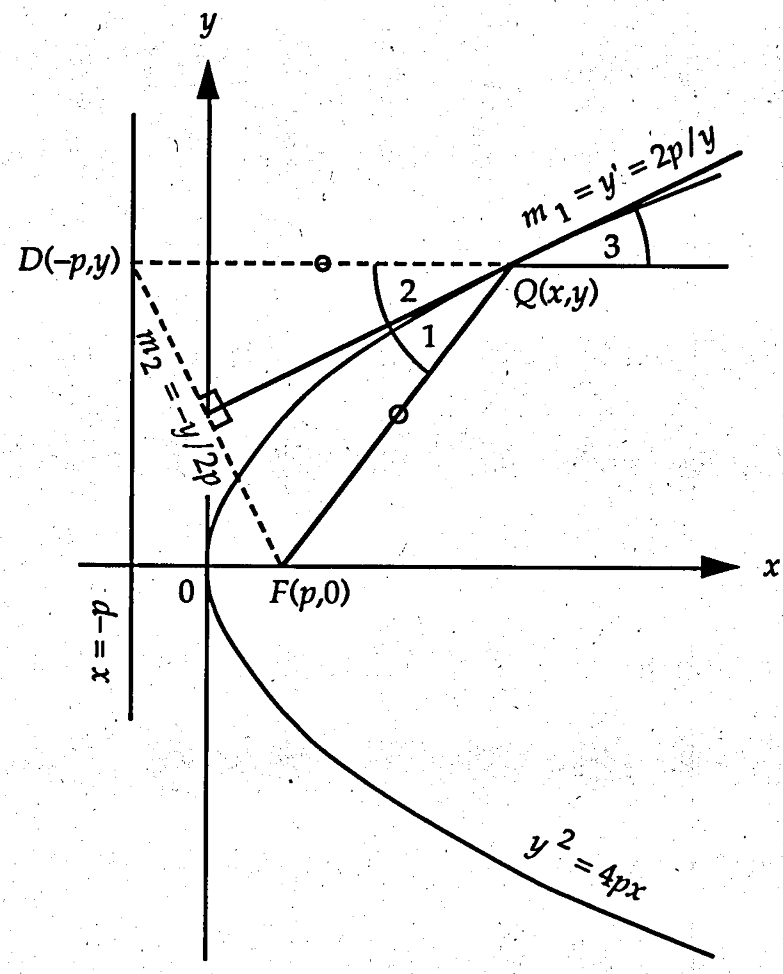
—Richard Courant

The Graphs of f and f^{-1} are Reflections about the Line $y = x$



—Ayoub B. Ayoub

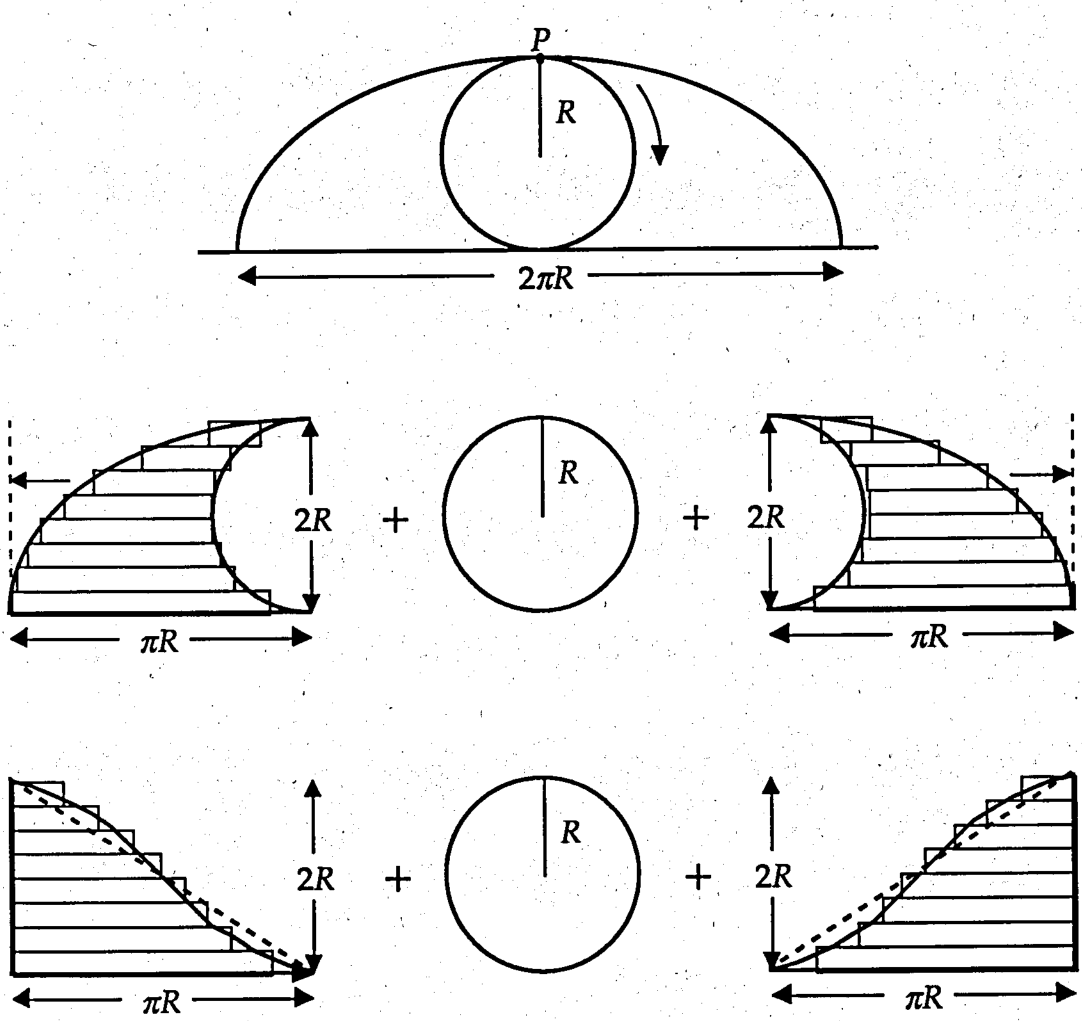
The Reflection Property of the Parabola



$$QF = QD \quad \& \quad m_1 \cdot m_2 = -1 \quad \Rightarrow \quad \angle 1 = \angle 2 = \angle 3$$

—Ayoub B. Ayoub

Area under an Arch of the Cycloid



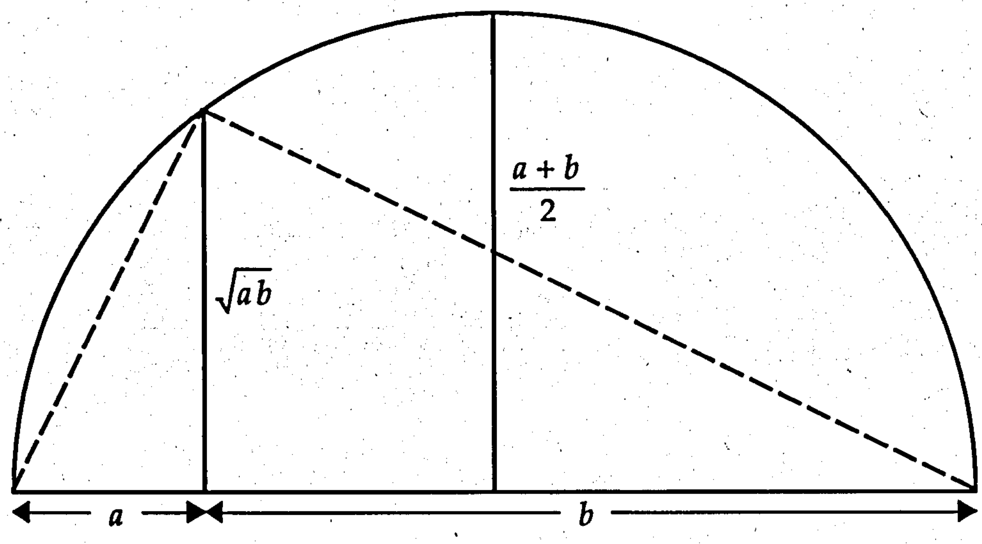
$$\frac{1}{2}\pi R \cdot 2R \quad + \quad \pi R^2 \quad + \quad \frac{1}{2}\pi R \cdot 2R$$

$$\Rightarrow A = 3\pi R^2$$

—Richard M. Beekman

Inequalities

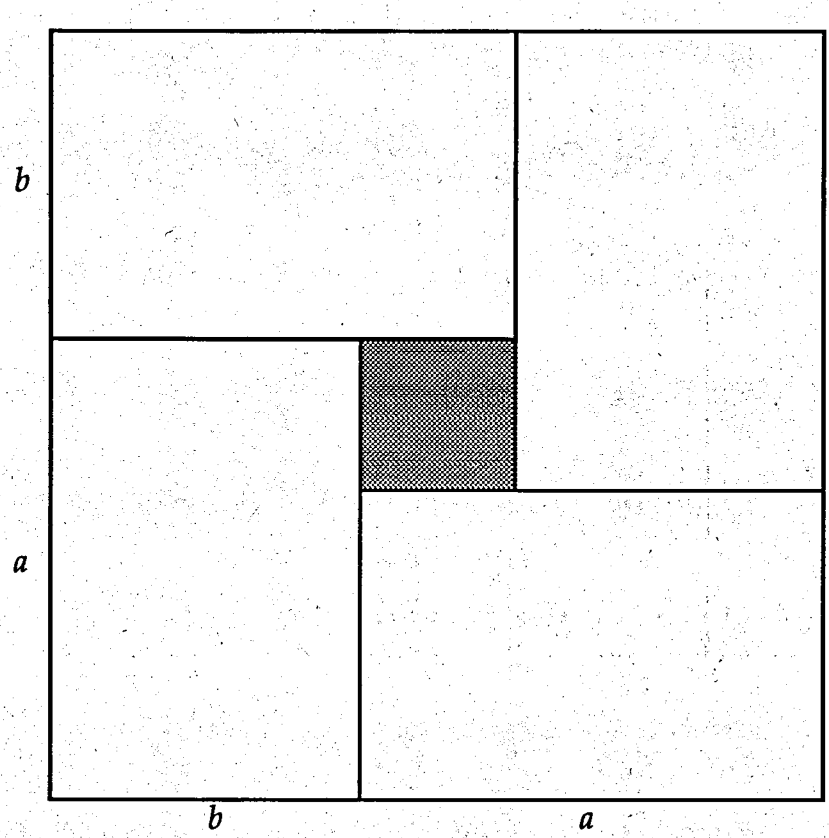
The Arithmetic Mean—Geometric Mean Inequality I



$$\sqrt{ab} \leq \frac{a+b}{2}$$

—Charles D. Gallant

The Arithmetic Mean—Geometric Mean Inequality II



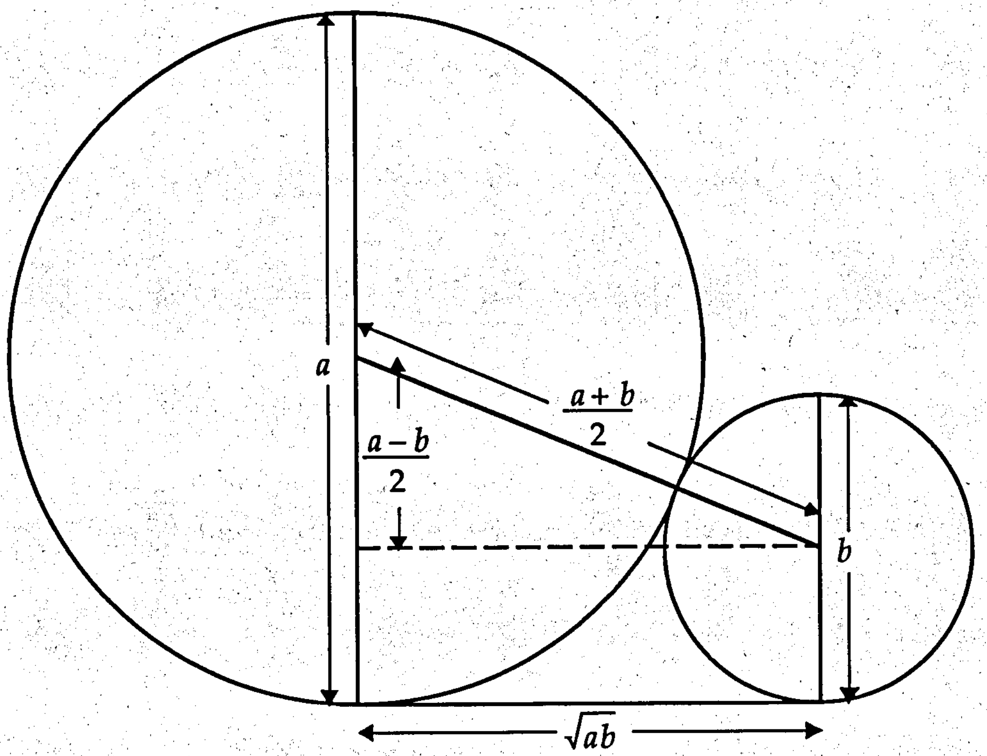
$$(a+b)^2 - (a-b)^2 = 4ab$$

$$\frac{a+b}{2} \geq \sqrt{ab}$$

—Doris Schattschneider

The Arithmetic Mean—Geometric Mean Inequality III

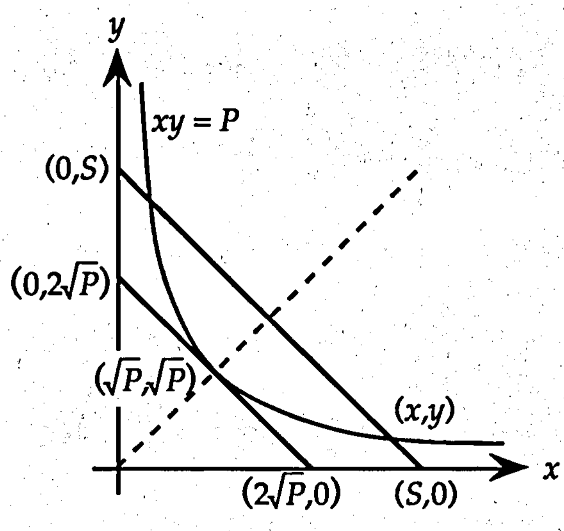
$$\frac{a+b}{2} \geq \sqrt{ab}, \text{ with equality if and only if } a = b$$



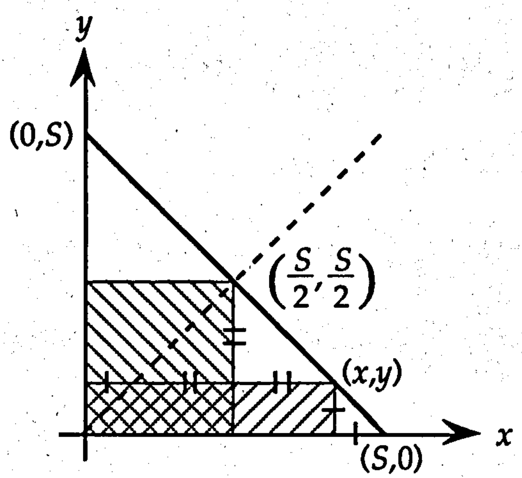
—Roland H. Eddy

Two Extremum Problems

For a given product, the sum of two positive numbers is minimal when the numbers are equal.

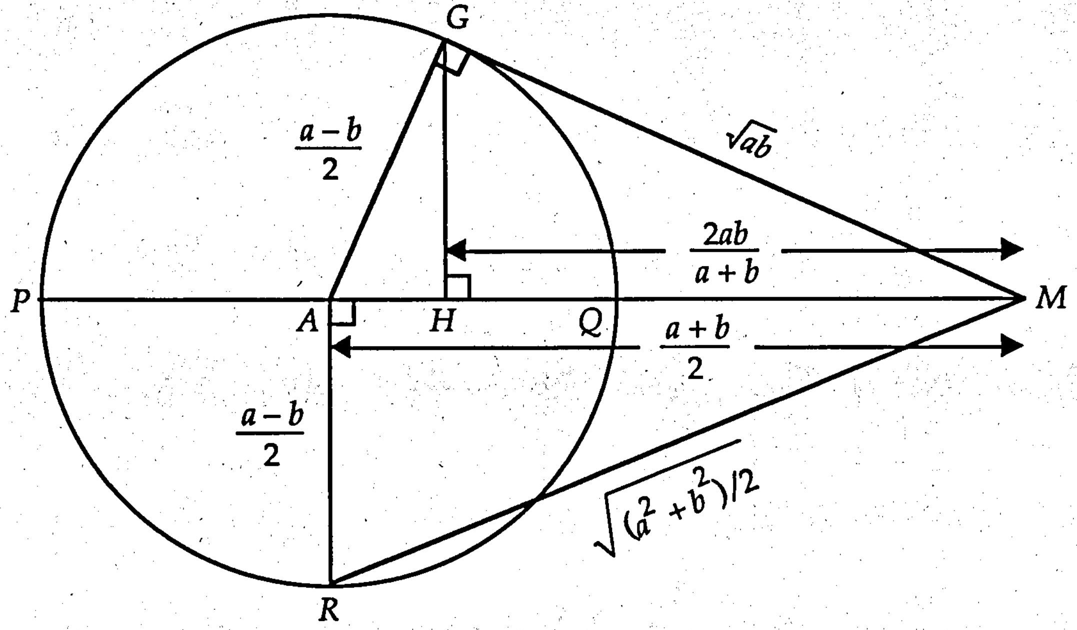


For a given sum, the product of two positive numbers is maximal when the numbers are equal.



—Paolo Montuchi and Warren Page

The Harmonic Mean—Geometric Mean—Arithmetic Mean—Root Mean Square Inequality I



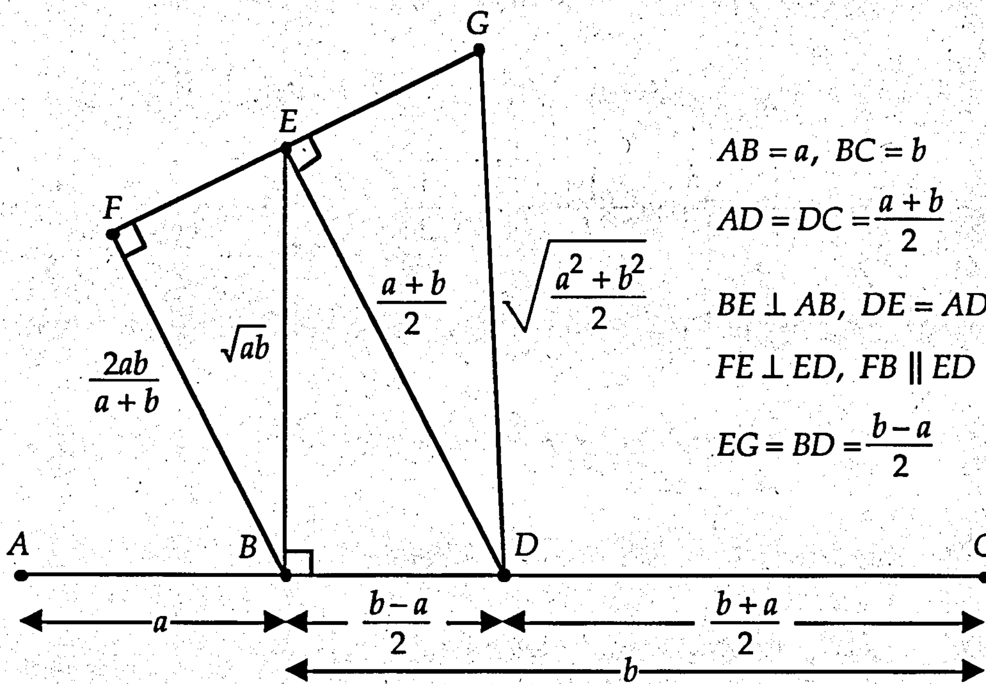
$$PM = a,\ QM = b,\ a > b > 0$$

$$HM < GM < AM < RM$$

$$\frac{2ab}{a+b} < \sqrt{ab} < \frac{a+b}{2} < \sqrt{(a^2+b^2)/2}$$

—RBN

The Harmonic Mean—Geometric Mean—Arithmetic Mean—Root Mean Square Inequality II



$AB = a,\ BC = b$

$AD = DC = \frac{a+b}{2}$

$BE \perp AB,\ DE = AD$

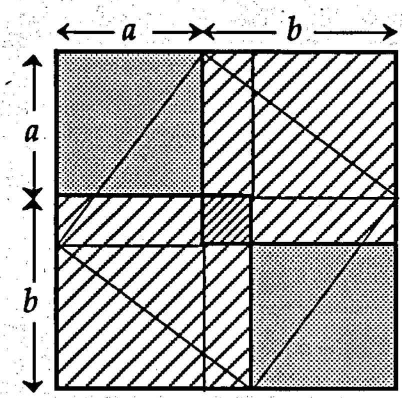
$FE \perp ED,\ FB \parallel ED$

$EG = BD = \frac{b-a}{2}$

—Sidney H. Kung

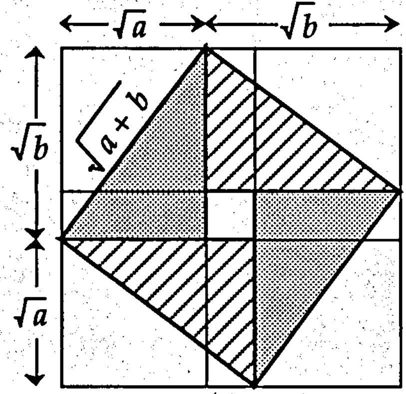
The Harmonic Mean—Geometric Mean—Arithmetic Mean—Root Mean Square Inequality III

$$a, b > 0 \Rightarrow \sqrt{(a^2 + b^2)/2} \geq \frac{a+b}{2} \geq \sqrt{ab} \geq \frac{2ab}{a+b}$$



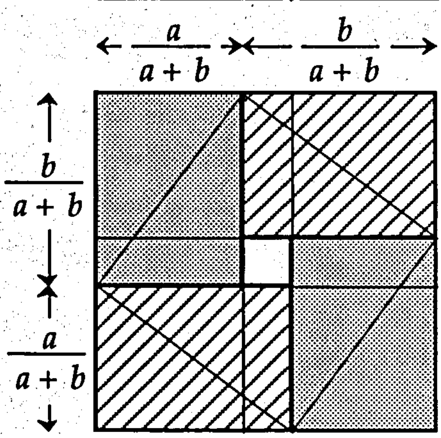
$$2a^2 + 2b^2 \geq (a+b)^2$$

$$\sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2}$$



$$(\sqrt{a+b})^2 \geq 4 \cdot \frac{1}{2}\sqrt{a}\sqrt{b}$$

$$\frac{a+b}{2} \geq \sqrt{ab}$$



$$1 \geq 4\frac{a}{a+b} \cdot \frac{b}{a+b}$$

$$\sqrt{ab} \geq \frac{2ab}{a+b}$$

—RBN

Five Means — and Their Means

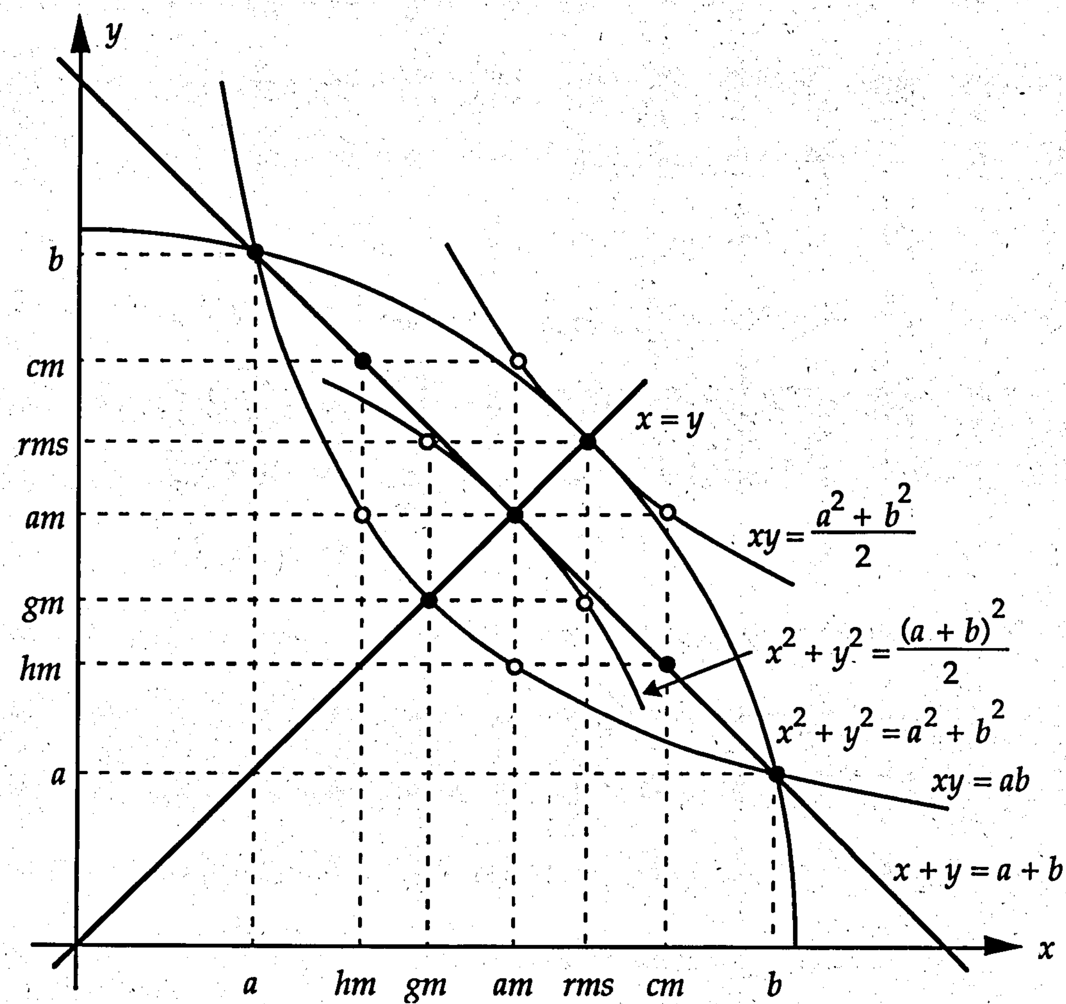
Arithmetic: $am = AM(a,b) = \dfrac{a+b}{2}$

Contraharmonic: $cm = CM(a,b) = \dfrac{a^2+b^2}{a+b}$

Geometric: $gm = GM(a,b) = \sqrt{ab}$

Harmonic: $hm = HM(a,b) = \dfrac{2ab}{a+b}$

Root Mean Square: $rms = RMS(a,b) = \sqrt{\dfrac{a^2+b^2}{2}}$



I. $0 < a < b \Rightarrow$

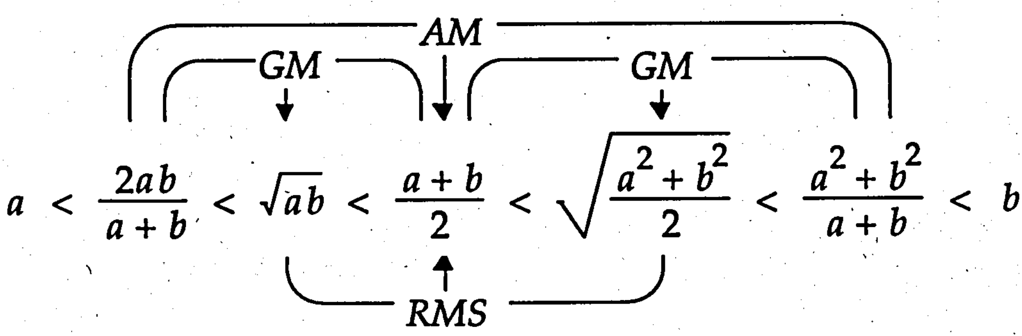
$$a < \frac{2ab}{a+b} < \sqrt{ab} < \frac{a+b}{2} < \sqrt{\frac{a^2+b^2}{2}} < \frac{a^2+b^2}{a+b} < b$$

II. $hm + cm = a + b \Rightarrow AM(hm, cm) = am.$

III. $hm \cdot am = a \cdot b \Rightarrow GM(hm, am) = gm.$

IV. $am \cdot cm = \dfrac{a^2+b^2}{2} \Rightarrow GM(am, cm) = rms.$

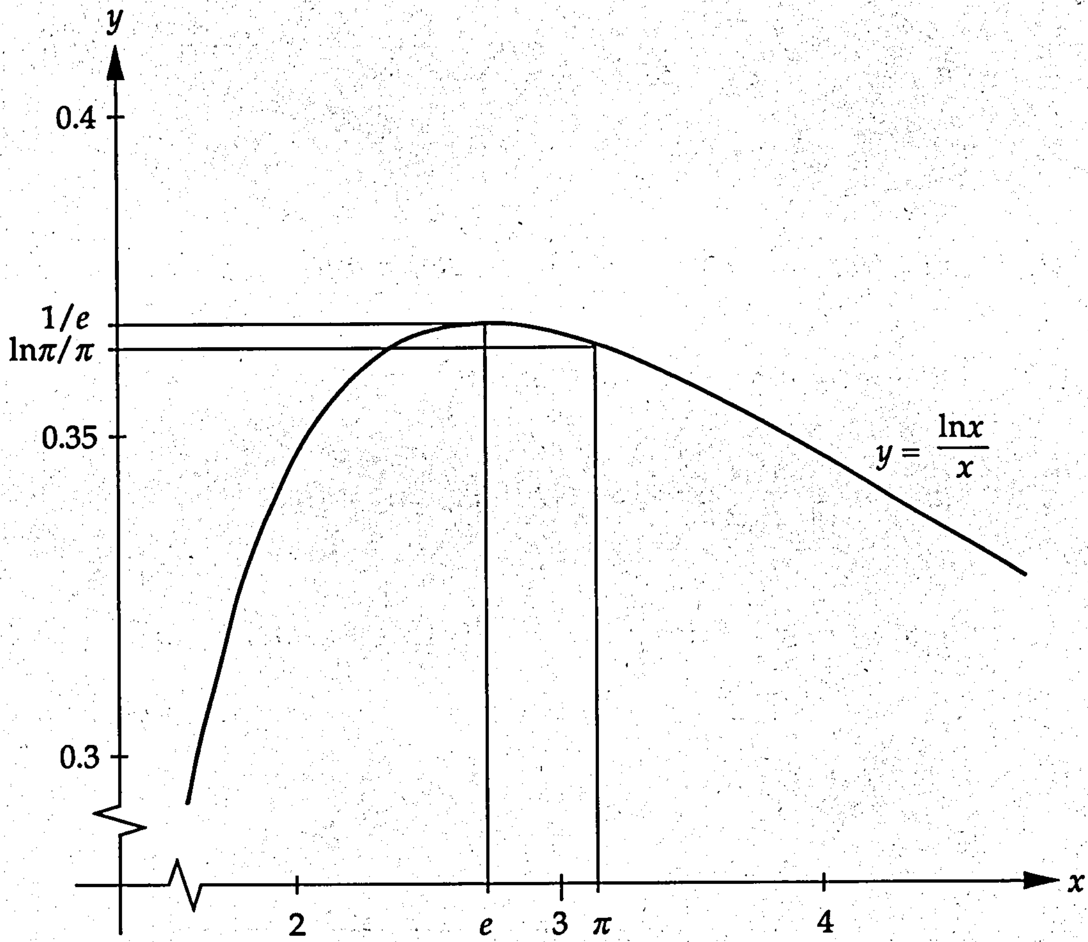
V. $gm^2 + rms^2 = \dfrac{(a+b)^2}{2} \Rightarrow RMS(gm, rms) = am.$



$$a < \frac{2ab}{a+b} < \sqrt{ab} < \frac{a+b}{2} < \sqrt{\frac{a^2+b^2}{2}} < \frac{a^2+b^2}{a+b} < b$$

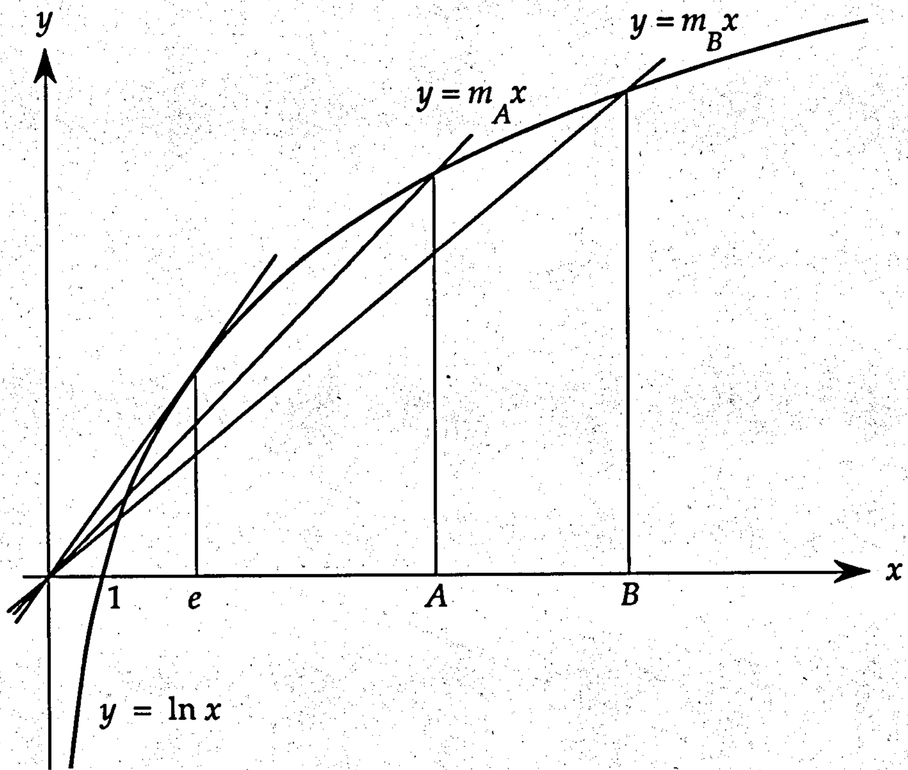
—RBN

$e^{\pi} > \pi^{e}$



—Fouad Nakhli

$A^B > B^A$ for $e \leq A < B$



$$e \leq A < B \Rightarrow m_A > m_B$$

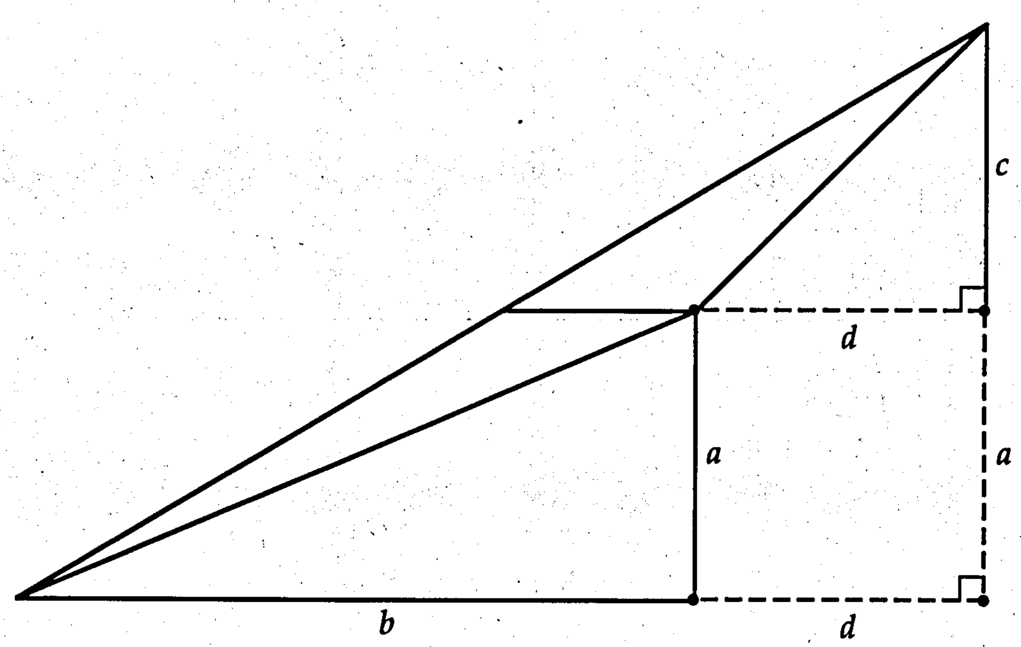
$$\Rightarrow \frac{\ln A}{A} > \frac{\ln B}{B}$$

$$\Rightarrow A^B > B^A$$

—Charles D. Gallant

The Mediant Property

$$\frac{a}{b} < \frac{c}{d} \Rightarrow \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$



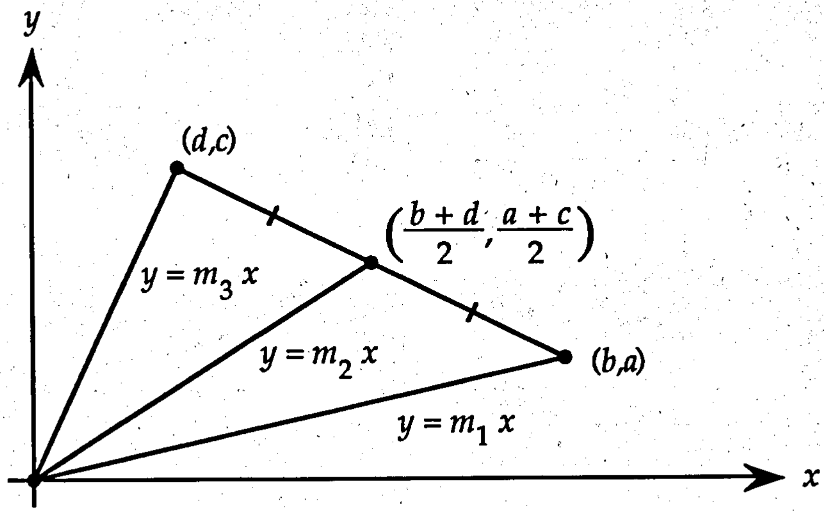
—Richard A. Gibbs

Regle des Nombres Moyens (Two Proofs)

[Nicolas Chuquet, *Le Triparty en la Science des Nombres*, 1484]

$$a, b, c, d > 0;\ \frac{a}{b} < \frac{c}{d} \Rightarrow \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

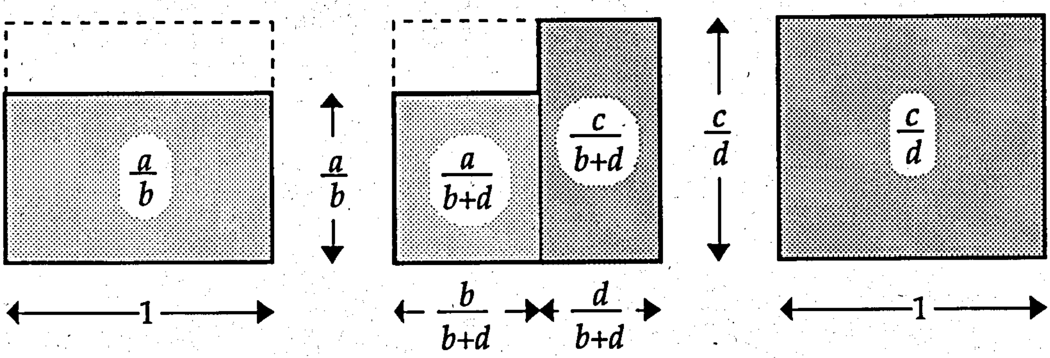
I.



$$m_1 < m_3 \Rightarrow m_1 < m_2 < m_3$$

—Li Changming

II.

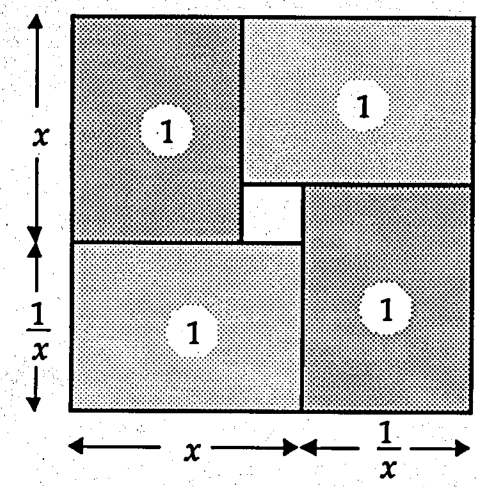


$$\frac{a}{b} < \frac{a}{b+d} + \frac{c}{b+d} < \frac{c}{d}$$

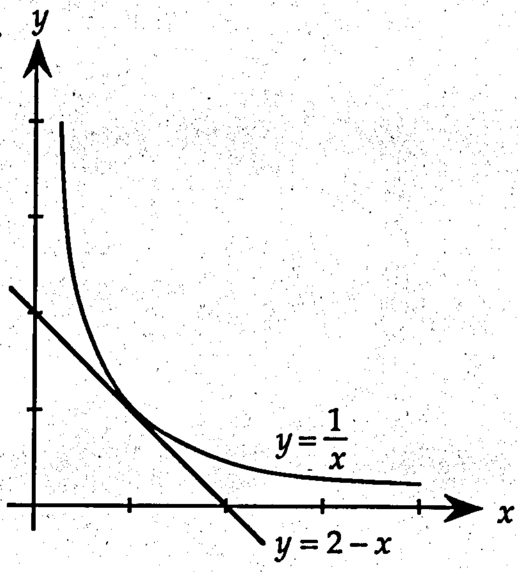
—RBN

The Sum of a Positive Number and its Reciprocal is at least Two (Four Proofs)

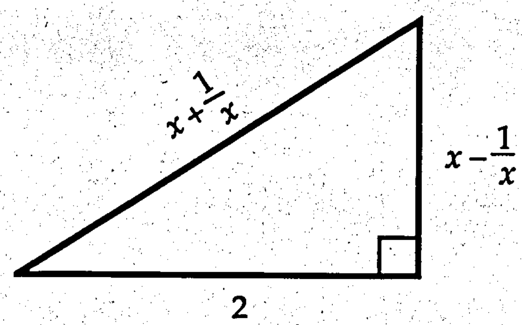
I.



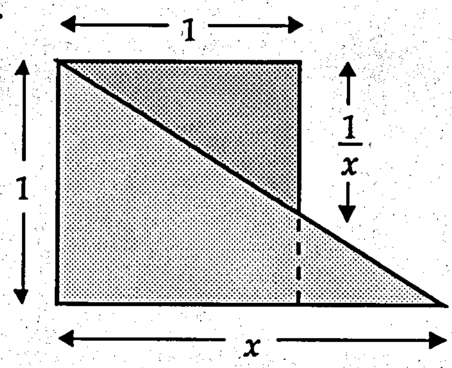
II.



III.



IV.

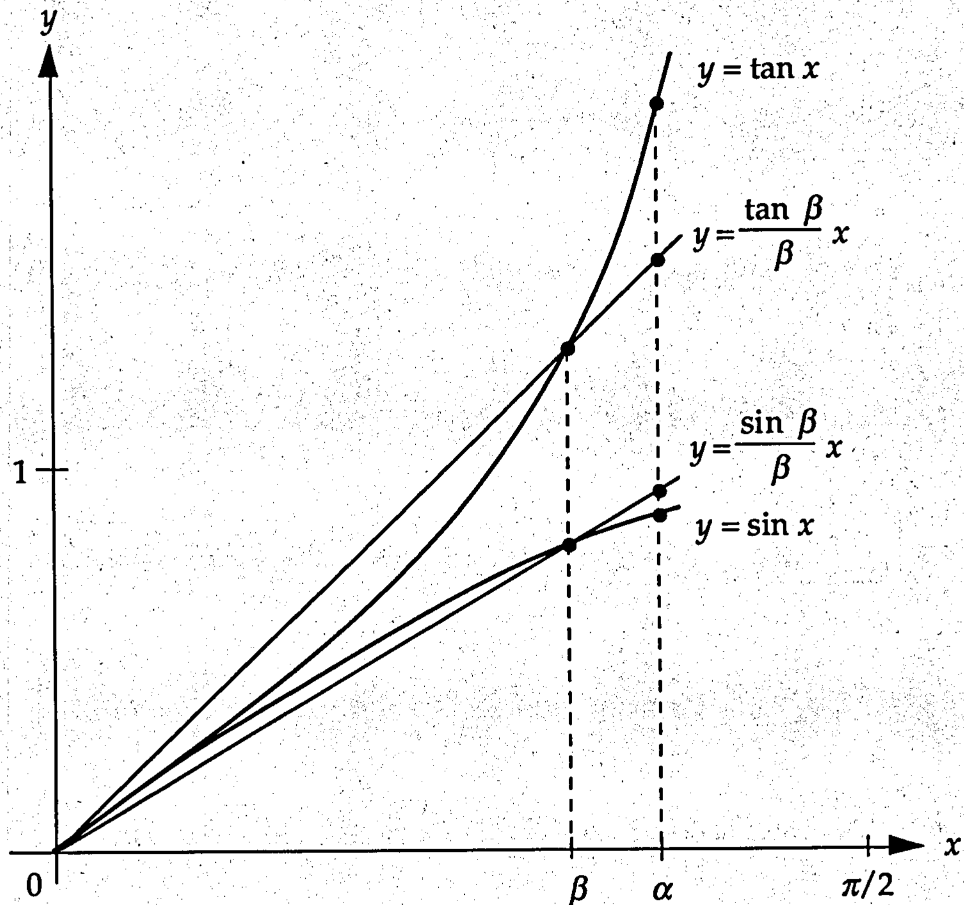


$$x \geq 1 \Rightarrow x + \frac{1}{x} \geq 2$$

—RBN

Aristarchus' Inequalities

$$0 < \beta < \alpha < \frac{\pi}{2} \Rightarrow \frac{\sin\alpha}{\sin\beta} < \frac{\alpha}{\beta} < \frac{\tan\alpha}{\tan\beta}$$



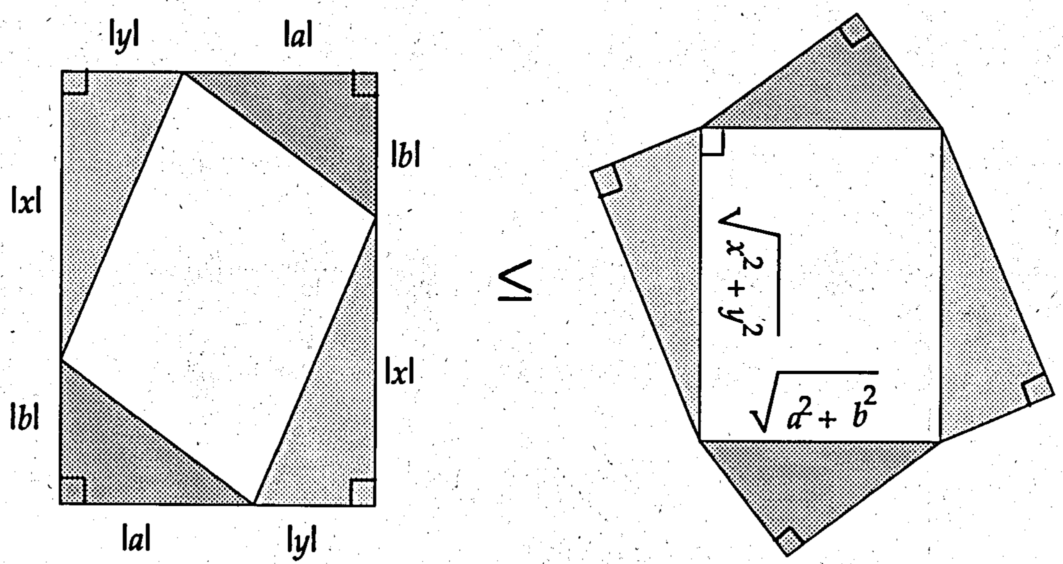
$$\sin\alpha < \frac{\sin\beta}{\beta}\alpha; \quad \frac{\tan\beta}{\beta}\alpha < \tan\alpha$$

$$\therefore \quad \frac{\sin\alpha}{\sin\beta} < \frac{\alpha}{\beta} < \frac{\tan\alpha}{\tan\beta}$$

—RBN

The Cauchy-Schwarz Inequality

$$|\langle a,b\rangle\cdot\langle x,y\rangle| \leq ||\langle a,b\rangle||\ ||\langle x,y\rangle||$$



$$(|a| + |y|)(|b| + |x|) \leq 2(\tfrac{1}{2}|a||b| + \tfrac{1}{2}|x||y|) + \sqrt{a^2+b^2}\sqrt{x^2+y^2}$$

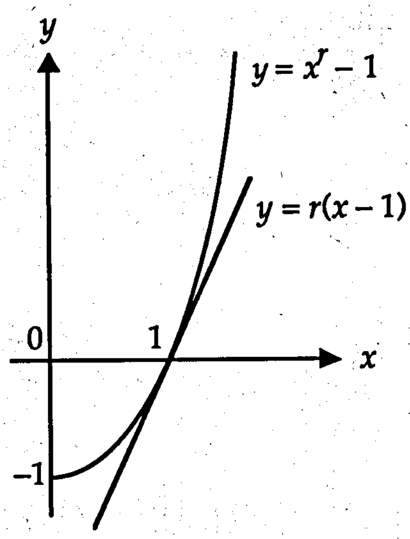
$$\therefore\ |ax + by| \leq |a||x| + |b||y| \leq \sqrt{a^2+b^2}\sqrt{x^2+y^2}$$

—RBN

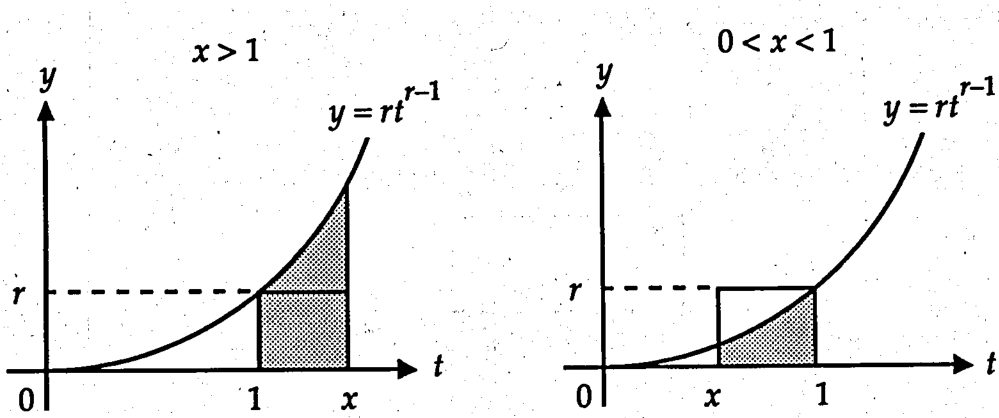
Bernoulli's Inequality (two proofs)

$$x > 0, x \neq 1, r > 1 \Rightarrow x^r - 1 > r(x-1)$$

I. (first semester calculus)



II. (second semester calculus)



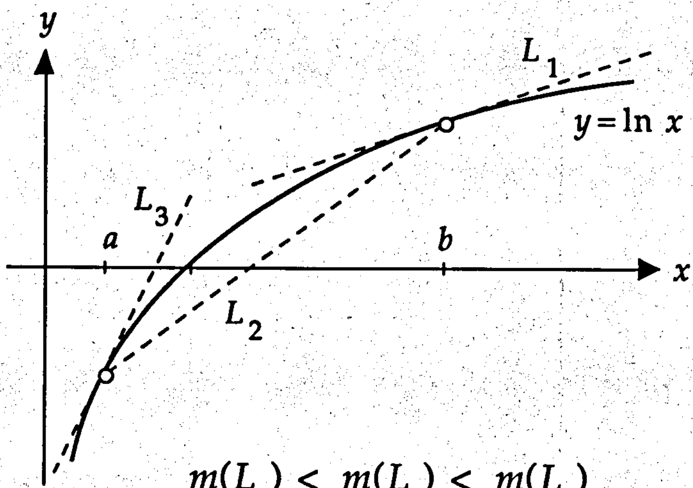
$$x^r - 1 = \int_1^x rt^{r-1}dt > r(x-1) \qquad 1 - x^r = \int_x^1 rt^{r-1}dt < r(1-x)$$

—RBN

Napier's Inequality (two proofs)

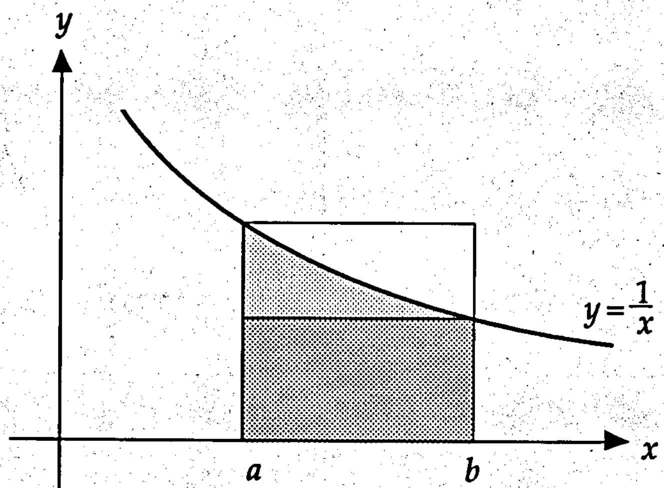
$$b > a > 0 \Rightarrow \frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a}$$

I. (first semester calculus)



$$m(L_1) < m(L_2) < m(L_3)$$

II. (second semester calculus)

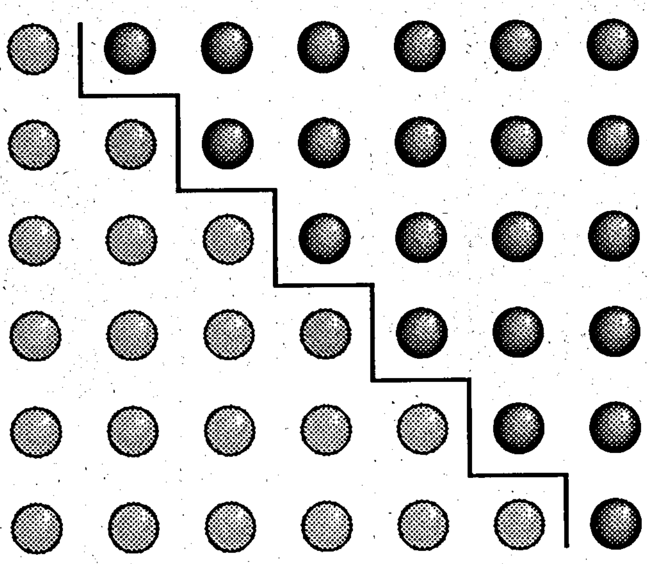


$$\frac{1}{b}(b-a) < \int_a^b \frac{1}{x}\,dx < \frac{1}{a}(b-a)$$

—RBN

Integer Sums

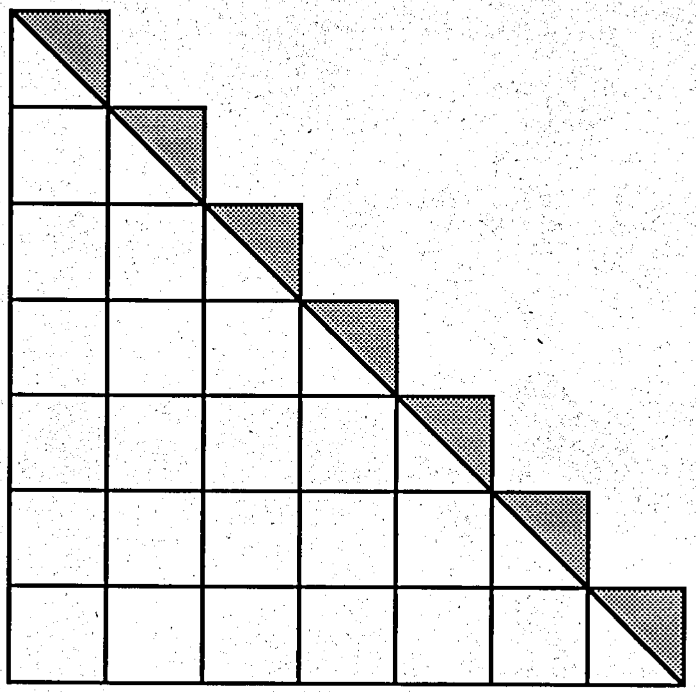
Sums of Integers I



$$1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$$

—"The ancient Greeks"
(as cited by Martin Gardner)

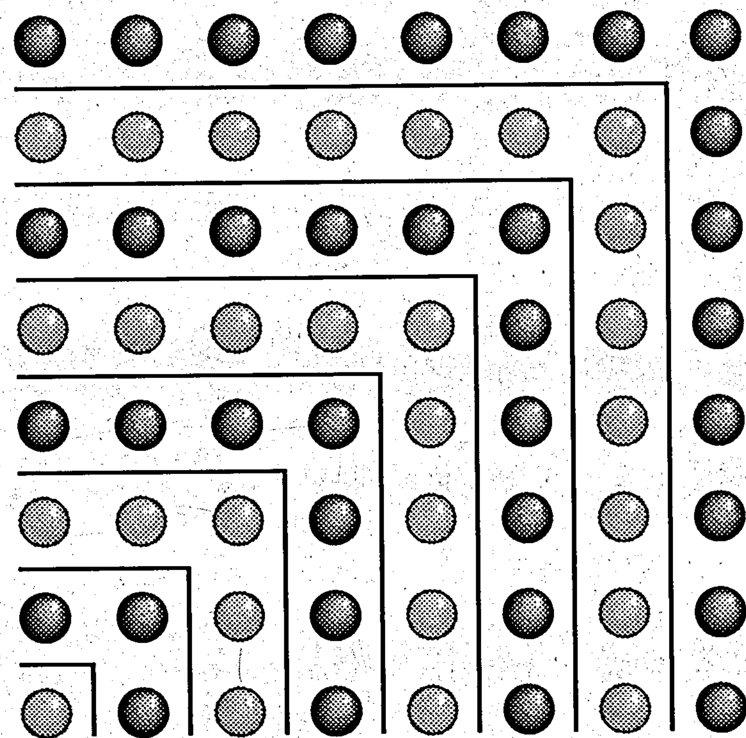
Sums of Integers II



$$1+2+\cdots+n=\frac{n^2}{2}+\frac{n}{2}$$

—Ian Richards

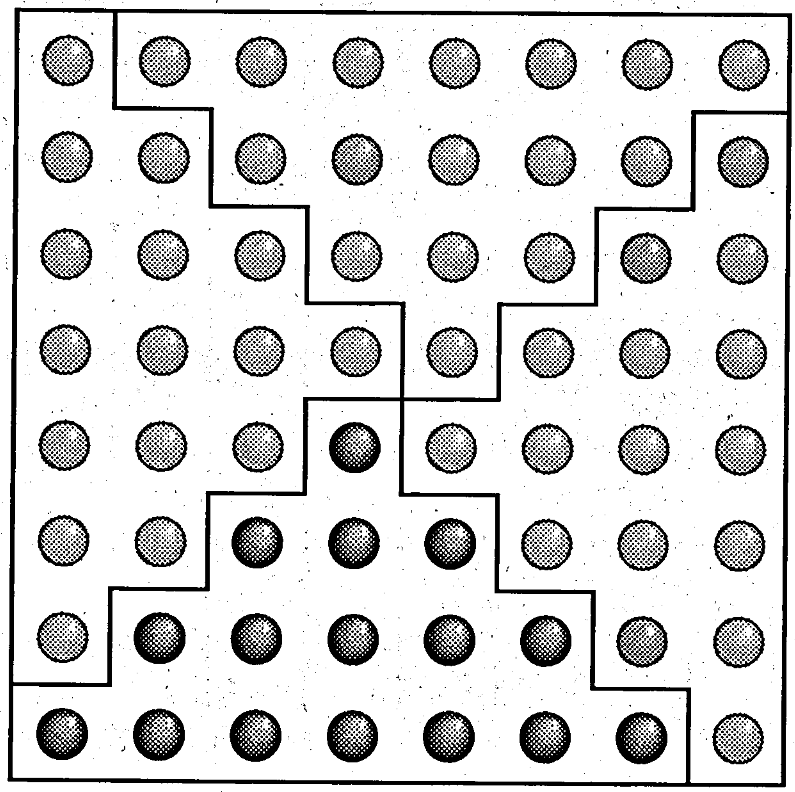
Sums of Odd Integers I



$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

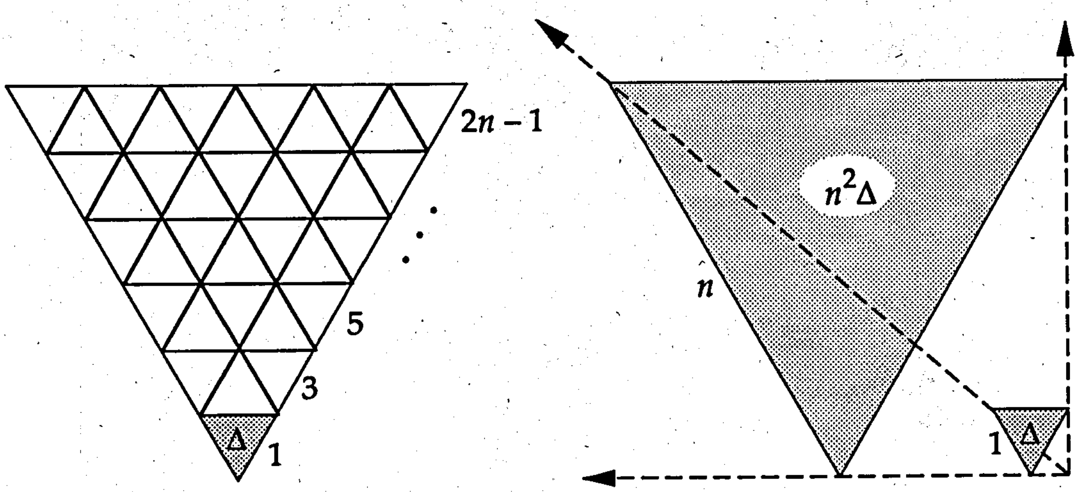
—Nicomachus of Gerasa (circa A. D. 100)

Sums of Odd Integers II



$$1 + 3 + \cdots + (2n - 1) = \frac{1}{4}(2n)^2 = n^2$$

Sums of Odd Integers III



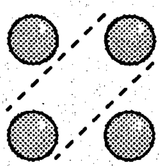
$$\Delta + 3\cdot\Delta + \cdots + (2n-1)\cdot\Delta = A = n^2\cdot\Delta$$

$$\sum_{i=1}^{n} (2i-1) = n^2$$

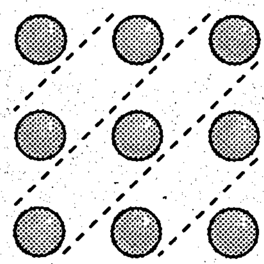
—Jenő Lehel

Squares and Sums of Integers

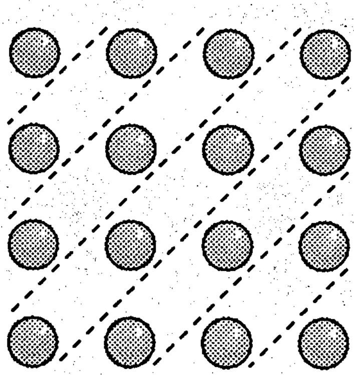
I.



$1+2+1=2^2$



$1+2+3+2+1 = 3^2$

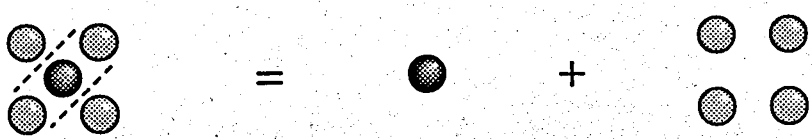


$1+2+3+4+3+2+1 = 4^2$

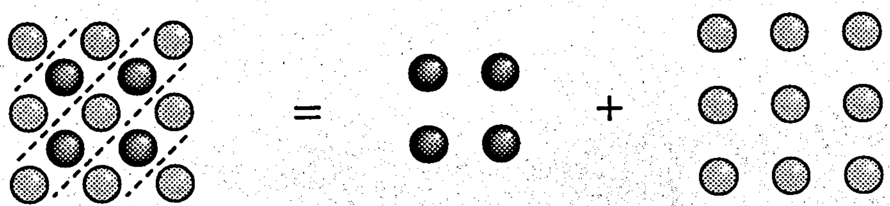
$$1+2+\cdots+(n-1)+n+(n-1)+\cdots+2+1 = n^2$$

—"The ancient Greeks"
(as cited by Martin Gardner)

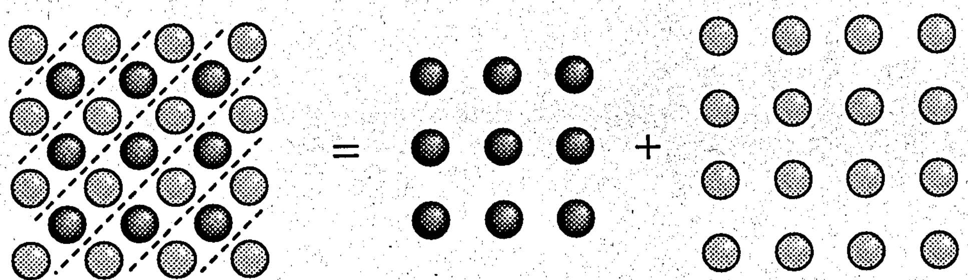
II.



$$1+3+1 = 1^2+2^2$$



$$1+3+5+3+1 = 2^2+3^2$$



$$1+3+5+7+5+3+1 = 3^2+4^2$$

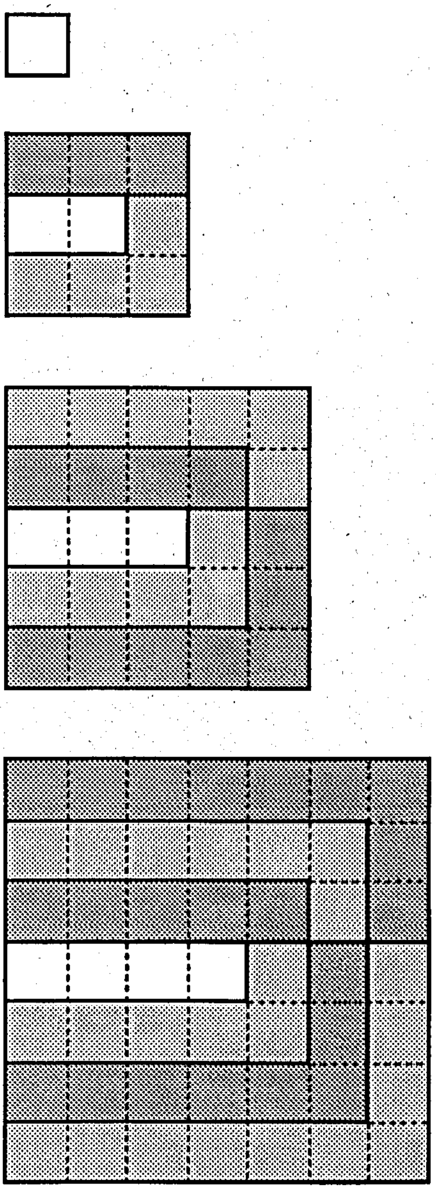
$$\vdots$$

$$1+3+\cdots+(2n-1)+(2n+1)+(2n-1)+\cdots+3+1 = n^2+(n+1)^2$$

—Hee Sik Kim

Arithmetic Progressions with Sum Equal to the Square of the Number of Terms

$$\sum_{k=n}^{3n-2} k = (2n-1)^2; \; n = 1, 2, 3, \cdots .$$



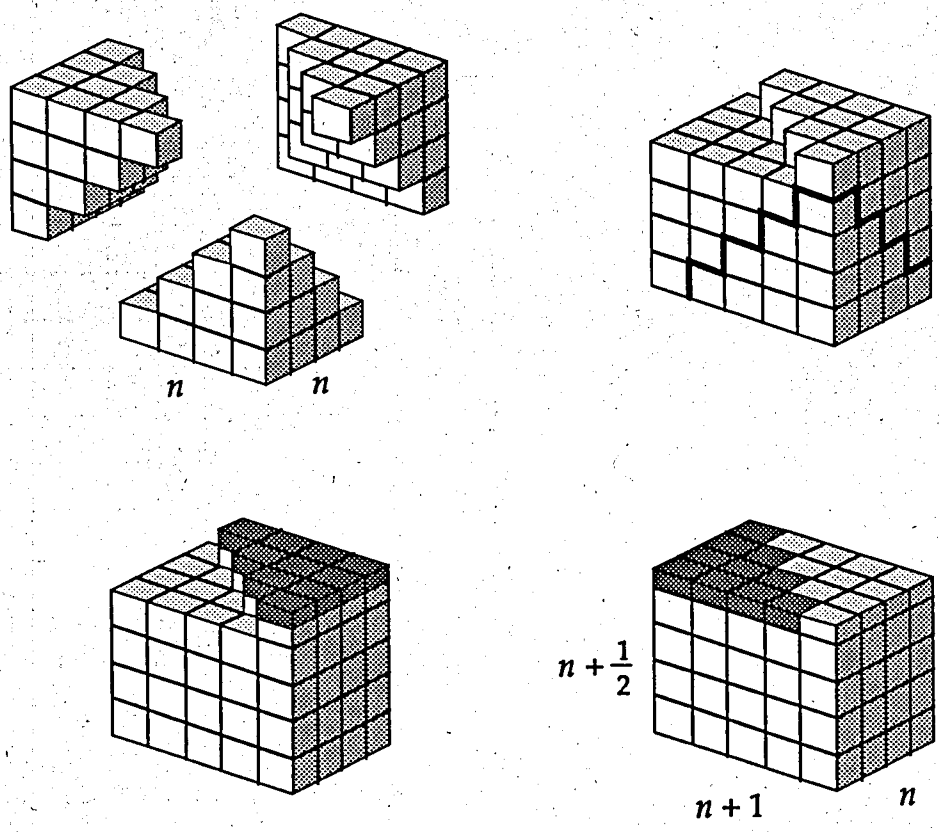
$n = 4$

$4 + 5 + 6 + 7 + 8 + 9 + 10 = 7^2$

—James O. Chilaka

Sums of Squares I

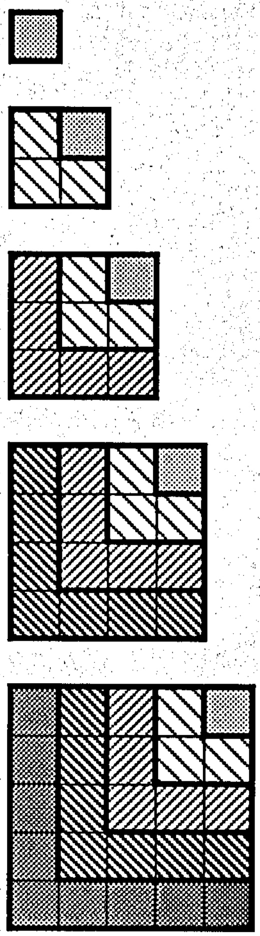
$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{3}n(n+1)(n+\frac{1}{2})$$

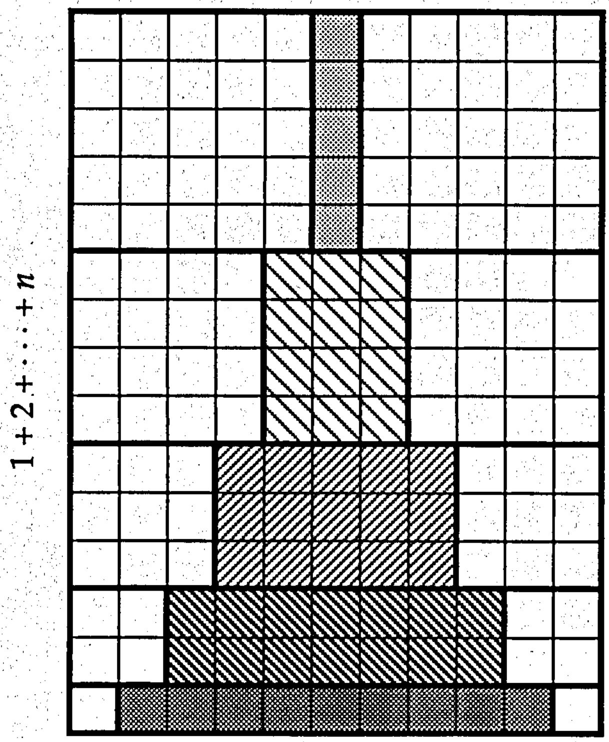


—Man-Keung Siu

Sums of Squares II

$$3(1^2 + 2^2 + \cdots + n^2) = (2n + 1)(1 + 2 + \cdots + n)$$





—Martin Gardner and Dan Kalman
(independently)

Sums of Squares III

$$3(1^2 + 2^2 + \cdots + n^2) = \frac{1}{2}n(n+1)(2n+1)$$

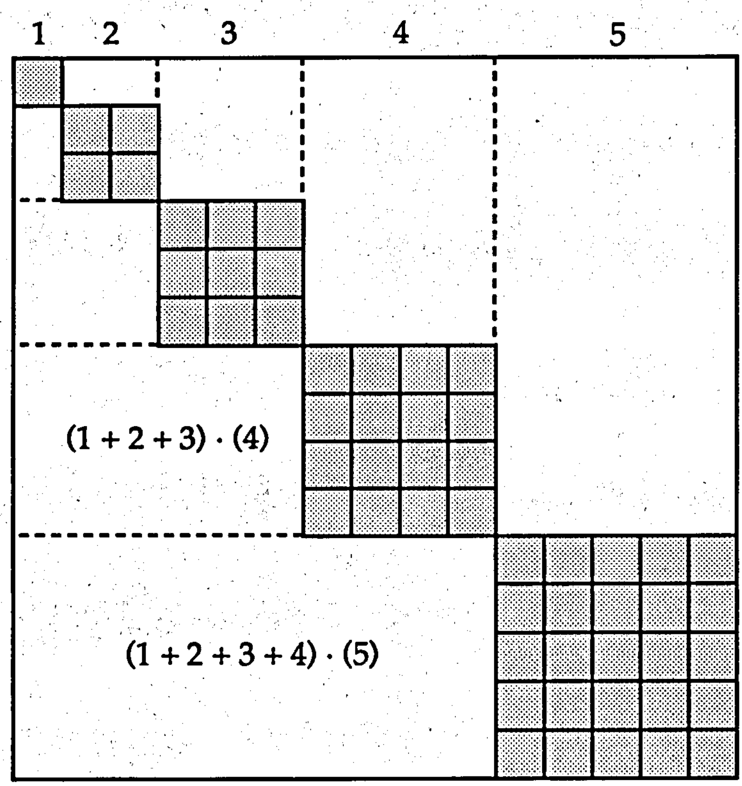
$$\begin{array}{ccccc} n & n & \cdots & n & n \\ n-1 & n-1 & \cdots & n-1 & \\ \vdots & \vdots & & & \\ 2 & 2 & & & \\ 1 & & & & \end{array} + \begin{array}{ccccc} n & n-1 & \cdots & 2 & 1 \\ n & n-1 & \cdots & 2 & \\ \vdots & \vdots & & & \\ n & n-1 & & & \\ n & & & & \end{array} + \begin{array}{ccccc} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & \\ \vdots & \vdots & & & \\ n-1 & n & & & \\ n & & & & \end{array}$$

$$= \begin{array}{ccccc} 2n+1 & 2n+1 & \cdots & 2n+1 & 2n+1 \\ 2n+1 & 2n+1 & \cdots & 2n+1 & \\ \vdots & \vdots & & & \\ 2n+1 & 2n+1 & & & \\ 2n+1 & & & & \end{array}$$

—Sidney H. Kung

Sums of Squares IV

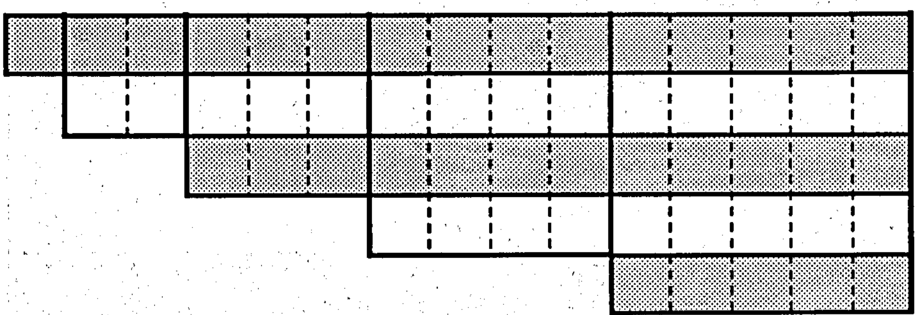
$$\sum_{k=1}^{n} k^2 = \left(\sum_{k=1}^{n} k\right)^2 - 2\sum_{k=1}^{n-1}\left[\left(\sum_{i=1}^{k} i\right)(k+1)\right]$$



—James O. Chilaka

Sums of Squares V

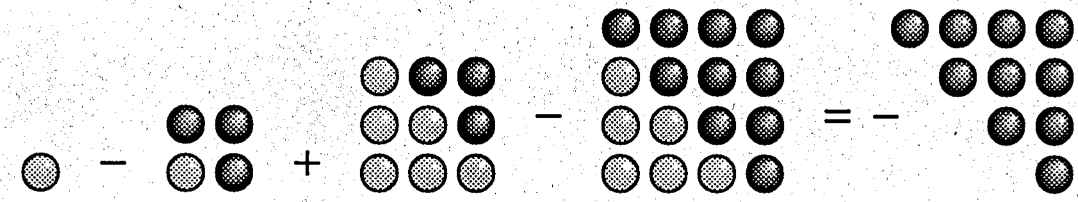
$$\sum_{i=1}^{n}\sum_{j=i}^{n} j = \sum_{i=1}^{n} i^2$$



—Pi-Chun Chuang

Alternating Sums of Squares

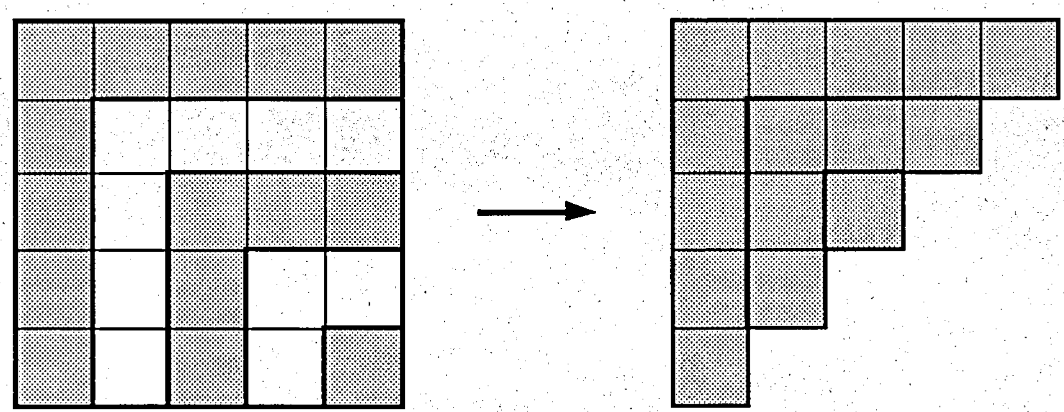
I.



$$\sum_{k=1}^{n} (-1)^{k+1} k^2 = (-1)^{n+1} T_n = (-1)^{n+1} \frac{n(n+1)}{2}$$

—Dave Logothetti

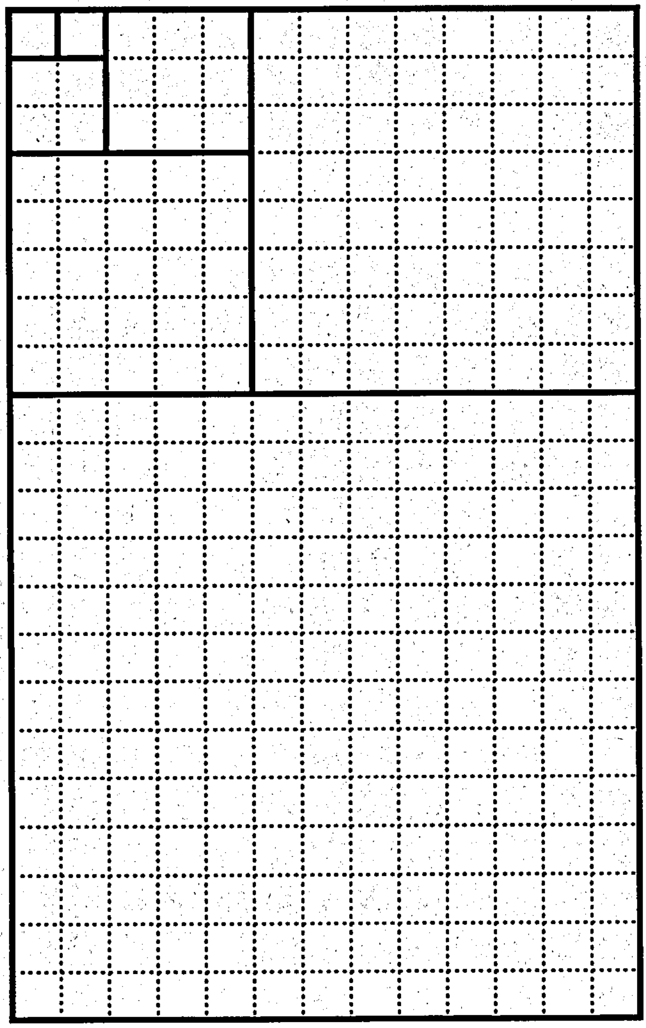
II.



$$n^2 - (n-1)^2 + \cdots + (-1)^{n-1}(1)^2 = \sum_{k=0}^{n} (-1)^k (n-k)^2 = \frac{n(n+1)}{2}$$

—Steven L. Snover

Sums of Squares of Fibonacci Numbers

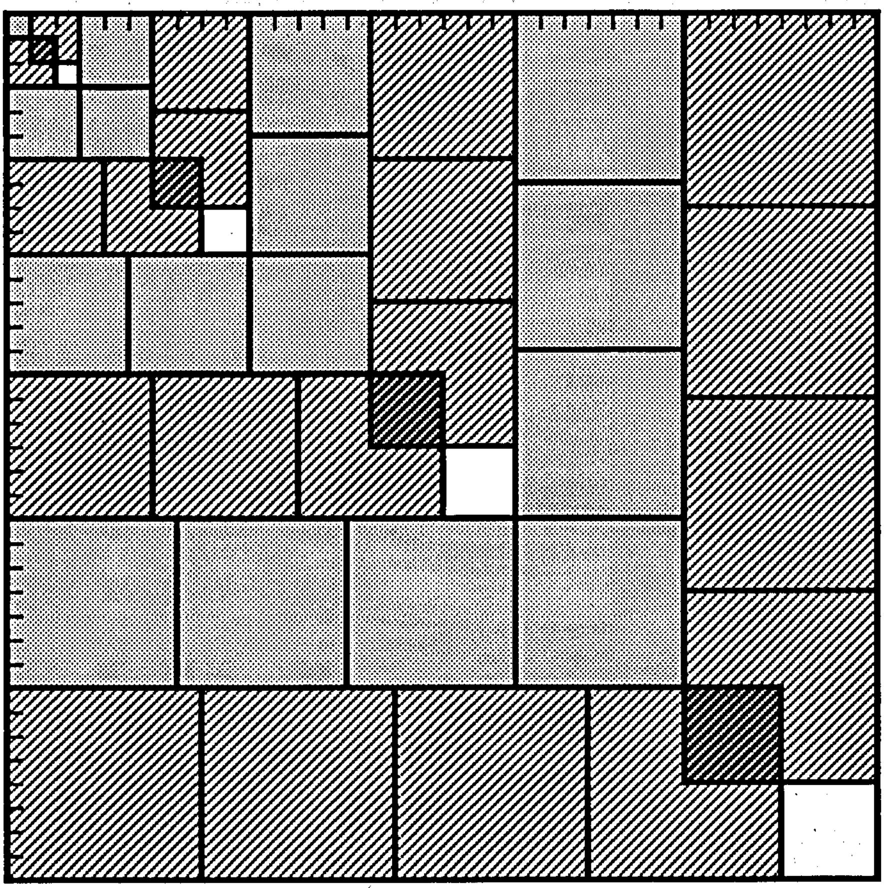


$$F_1 = F_2 = 1; F_{n+2} = F_{n+1} + F_n \Rightarrow F_1^2 + F_2^2 + \ldots + F_n^2 = F_n F_{n+1}$$

—Alfred Brousseau

Sums of Cubes I

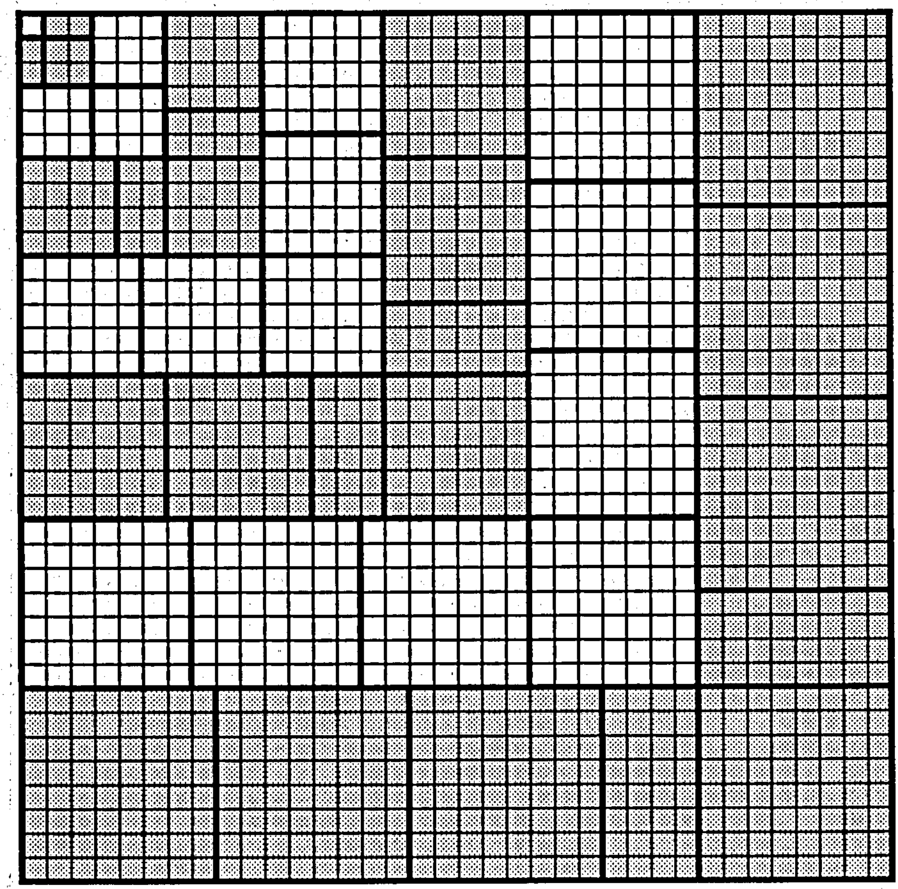
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$



—Solomon W. Golomb

Sums of Cubes II

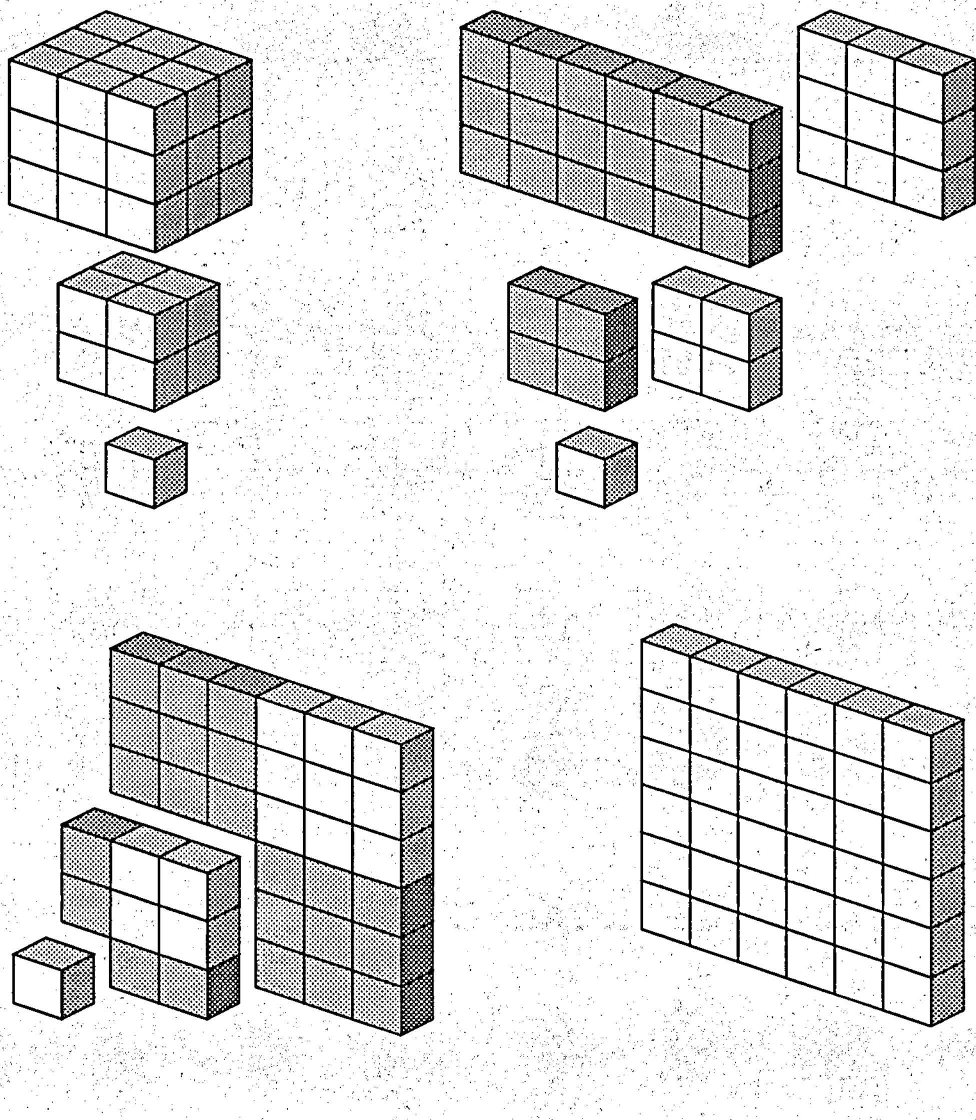
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$



—J. Barry Love

Sums of Cubes III

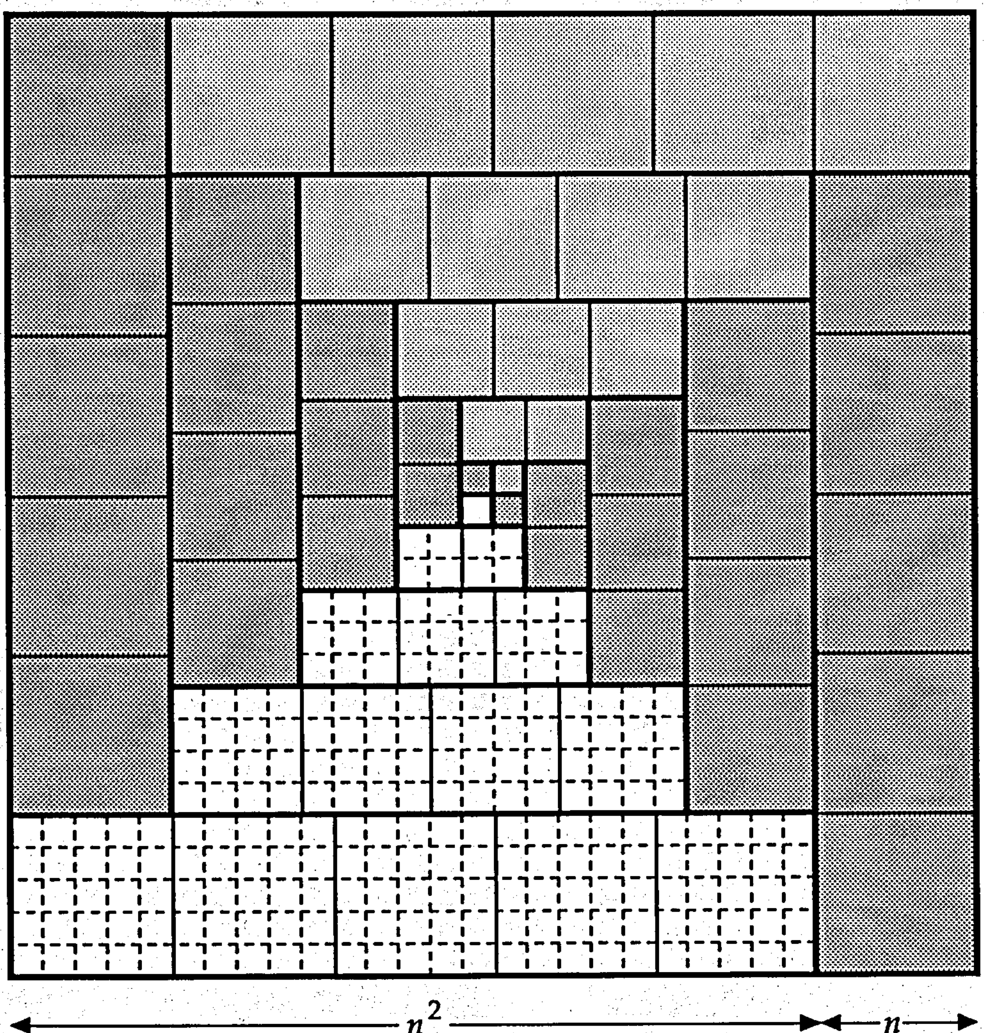
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$



—Alan L. Fry

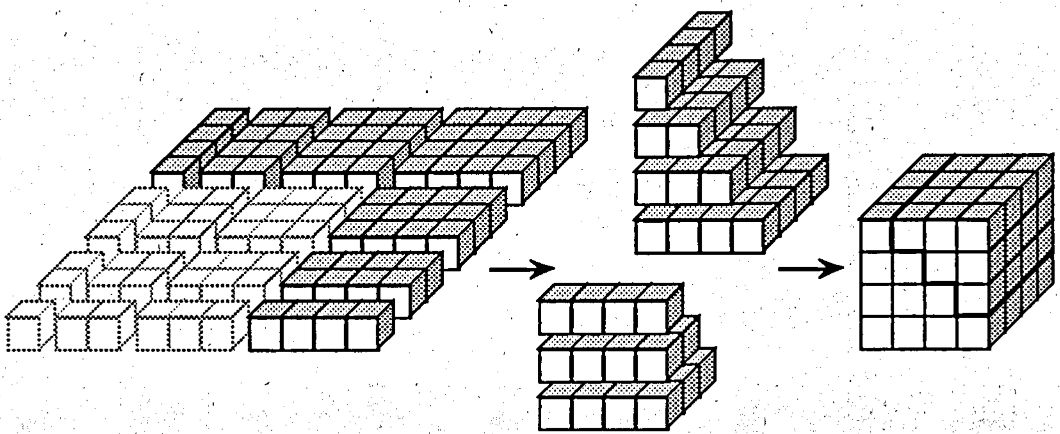
Sums of Cubes IV

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}[n(n+1)]^2$$

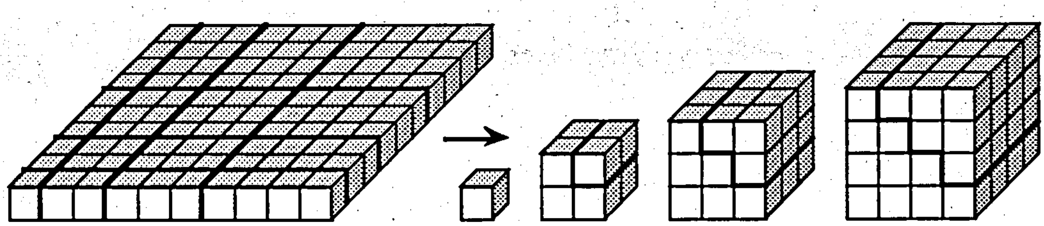


—Antonella Cupillari and Warren Lushbaugh
(independently)

Sum of Cubes V



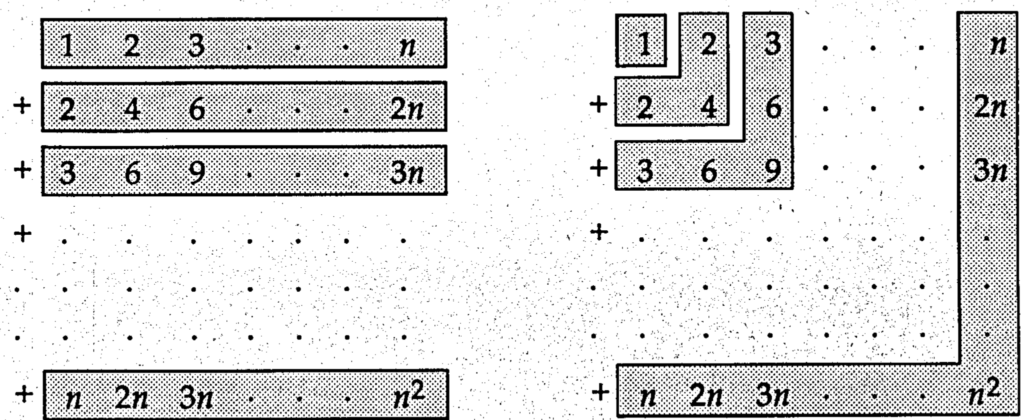
$$t_n = 1 + 2 + \cdots + n \;\Rightarrow\; t_n^2 - t_{n-1}^2 = n^3$$



$$t_n^2 = (1 + 2 + \cdots + n)^2 = 1^3 + 2^3 + 3^3 + \cdots + n^3$$

—RBN

Sum of Cubes VI



$$= \sum_{i=1}^{n} i + 2\sum_{i=1}^{n} i + \cdots + n\sum_{i=1}^{n} i \qquad = 1(1)^2 + 2(2)^2 + \cdots + n(n)^2$$

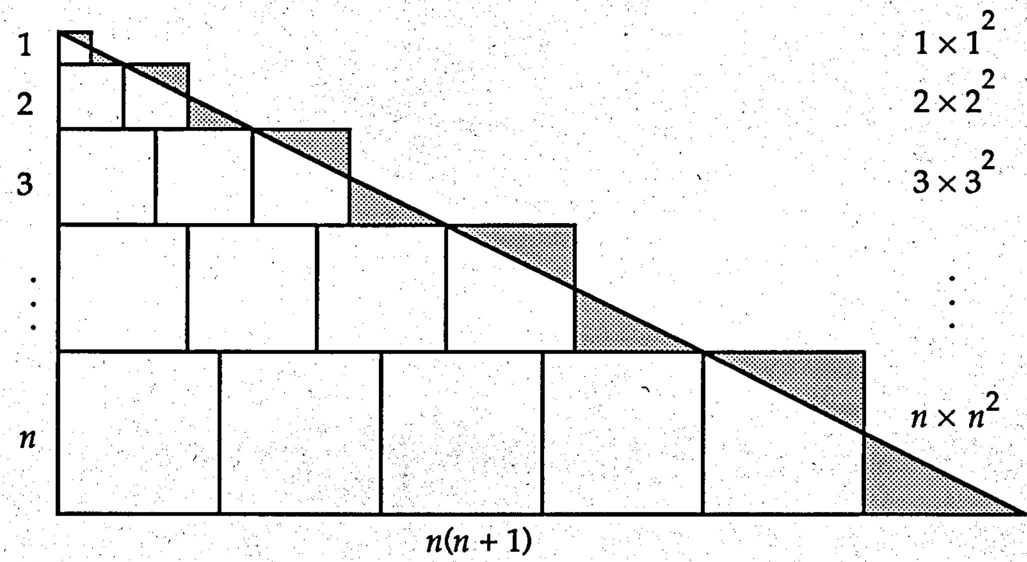
$$= \left(\sum_{i=1}^{n} i\right)^2 \qquad = \sum_{i=1}^{n} i^3$$

—Farhood Pouryoussefi

Sums of Integers and Sums of Cubes

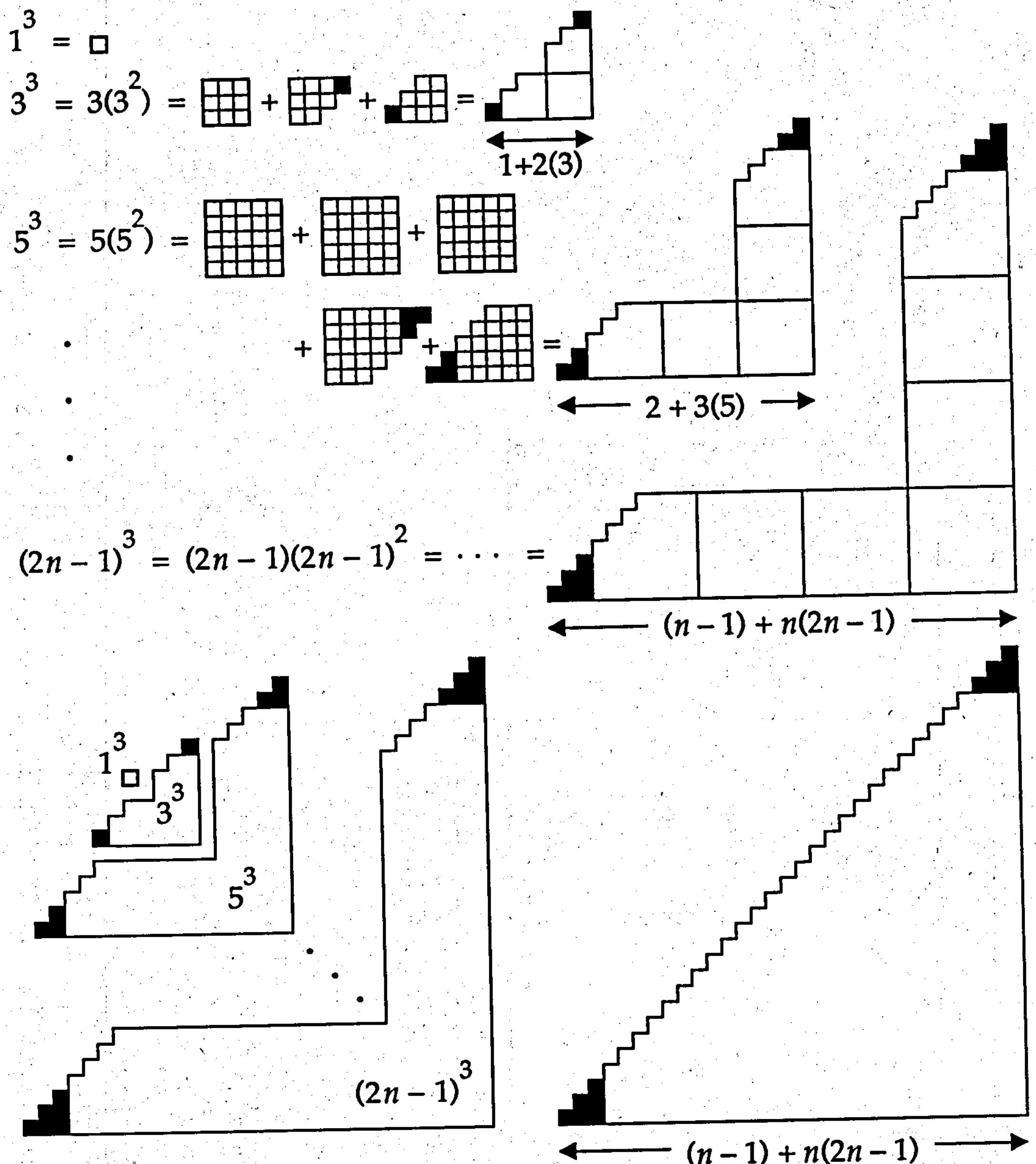
$$1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$$

$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{1}{2}n(n+1)\right)^2$$



—Georg Schrage

Sums of Odd Cubes are Triangular Numbers

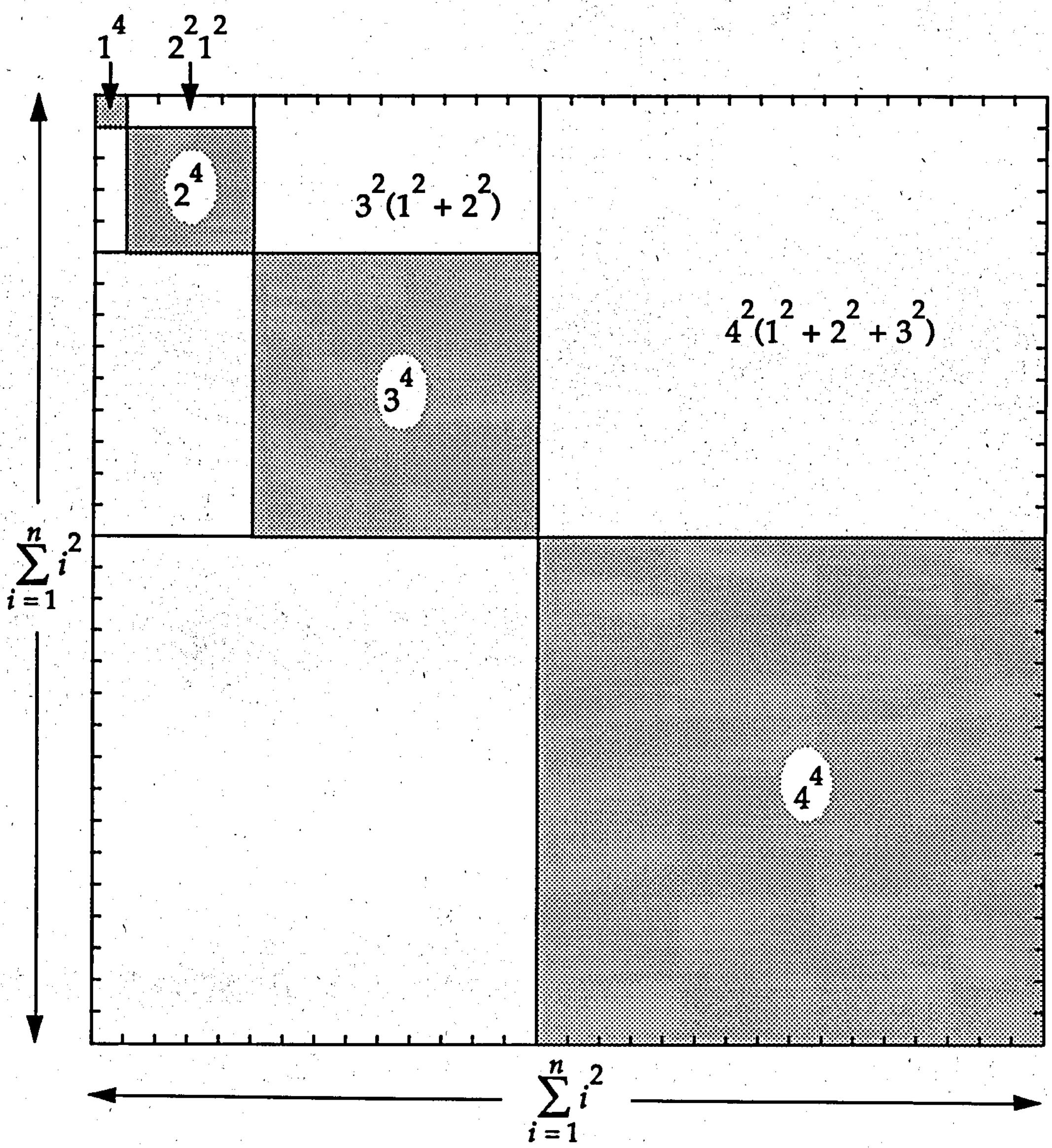


$$1^3+3^3+5^3+\cdots+(2n-1)^3 = 1+2+3+\cdots+(2n^2-1) = n^2(2n^2-1)$$

—Monte J. Zerger

Sums of Fourth Powers

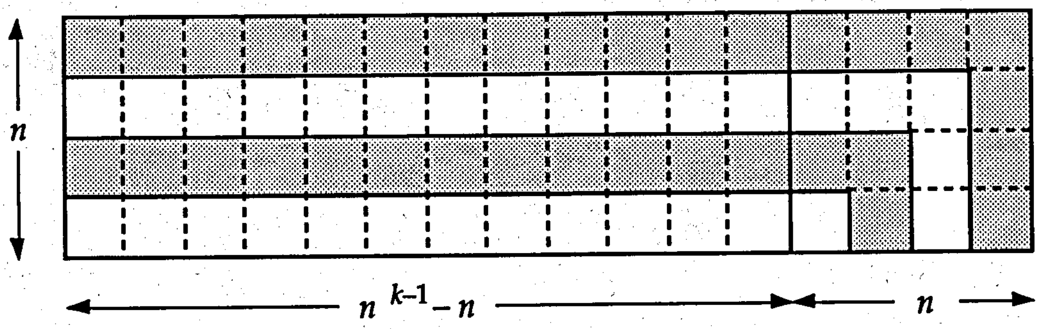
$$\sum_{i=1}^{n} i^4 = \left(\sum_{i=1}^{n} i^2\right)^2 - 2\left[\sum_{k=2}^{n}\left(k^2 \sum_{i=1}^{k-1} i^2\right)\right]$$



—Elizabeth M. Markham

kth Powers as Sums of Consecutive Odd Numbers

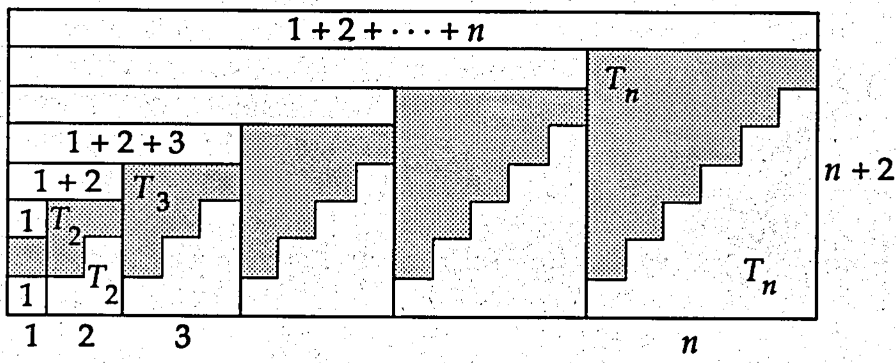
$$n^k = (n^{k-1} - n + 1) + (n^{k-1} - n + 3) + \cdots + (n^{k-1} - n + 2n - 1);$$
$$k = 2, 3, \cdots.$$



—N. Gopalakrishnan Nair

Sums of Triangular Numbers I

$$T_n = 1 + 2 + \cdots + n \Rightarrow T_1 + T_2 + \cdots + T_n = \frac{n(n+1)(n+2)}{6}$$



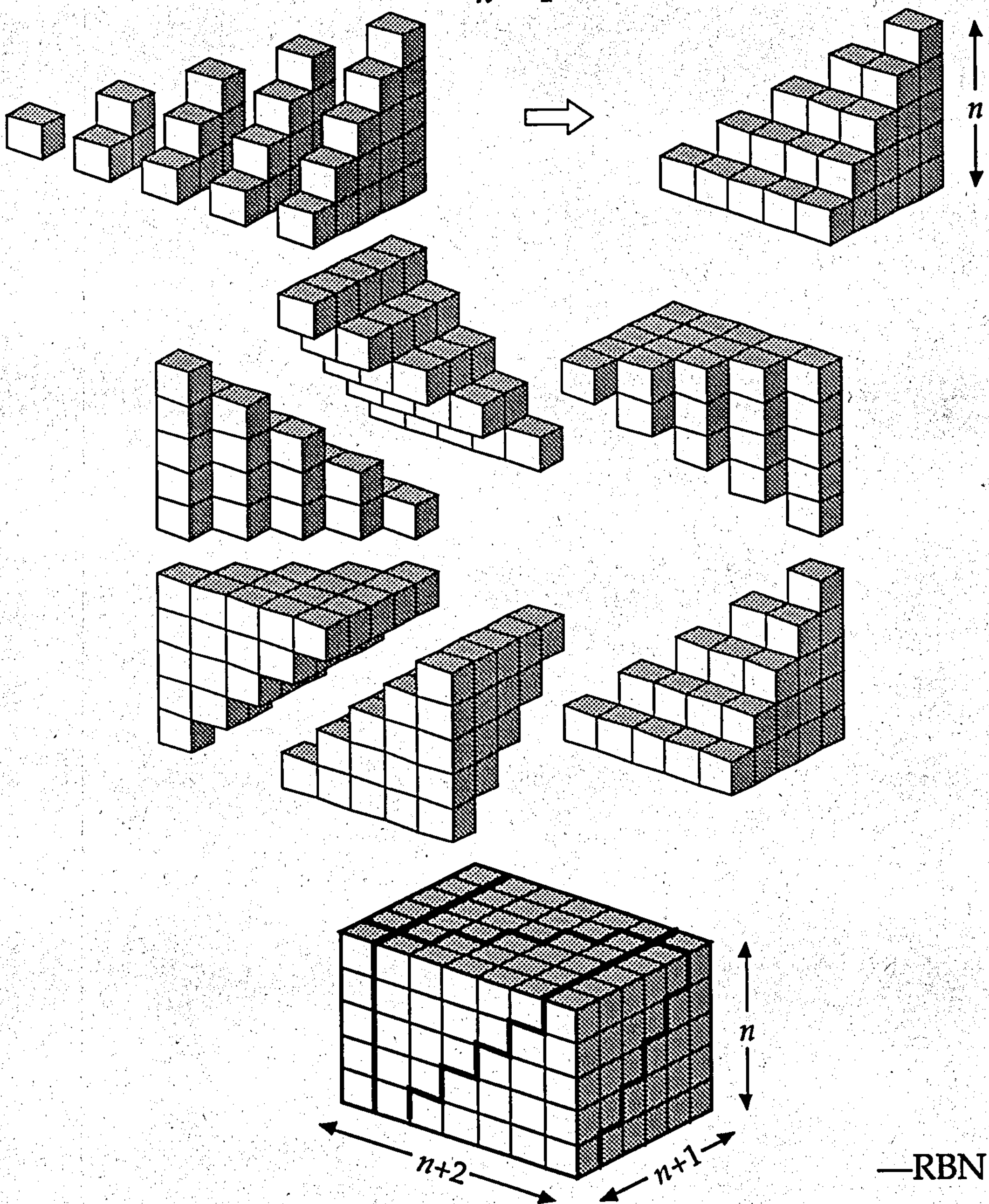
$$3(T_1 + T_2 + \cdots + T_n) = (n+2) \cdot T_n$$

$$T_1 + T_2 + \cdots + T_n = \frac{(n+2)}{3} \cdot \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$$

—Monte J. Zerger

Sums of Triangular Numbers II

$$T_k = 1 + 2 + \cdots + k \Rightarrow \sum_{k=1}^{n} T_k = \frac{1}{6}n(n+1)(n+2)$$



—RBN

Sums of Triangular Numbers III

$$T_k = 1 + 2 + \cdots + k \Rightarrow 3\sum_{k=1}^{n} T_k = \frac{1}{2}n(n+1)(n+2)$$

$$\begin{array}{llllll}
1 & & & & & \\
1 & 2 & & & & \\
1 & 2 & 3 & & & \\
\cdot & \cdot & \cdot & \cdot & & \\
\cdot & \cdot & \cdot & & \cdot & \\
1 & 2 & \cdots & n-1 & & \\
1 & 2 & \cdots & n-1 & n &
\end{array}
+
\begin{array}{lllll}
1 & & & & \\
2 & 1 & & & \\
3 & 2 & 1 & & \\
\cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & & \cdot \\
n-1 & n-2 & \cdots & 1 & \\
n & n-1 & \cdots & 2 & 1
\end{array}
+
\begin{array}{lllll}
n & & & & \\
n-1 & n-1 & & & \\
n-2 & n-2 & n-2 & & \\
\cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & & \cdot \\
2 & 2 & \cdots & 2 & \\
1 & 1 & \cdots & 1 & 1
\end{array}$$

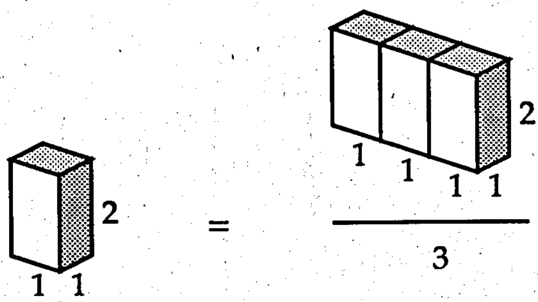
$$= \begin{array}{lllll}
n+2 & & & & \\
n+2 & n+2 & & & \\
n+2 & n+2 & n+2 & & \\
\cdot & \cdot & \cdot & & \\
\cdot & \cdot & \cdot & \cdot & \\
n+2 & n+2 & \cdots & n+2 & \\
n+2 & n+2 & \cdots & n+2 & n+2
\end{array}$$

$$3(T_1 + T_2 + \ldots + T_n) = T_n \cdot (n+2)$$

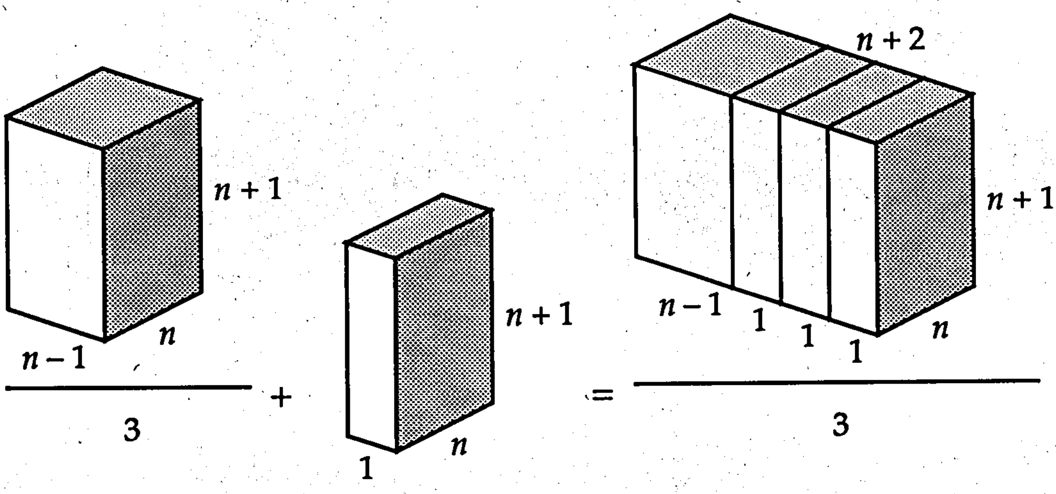
Sums of Oblong Numbers I

$$(1\times 2)+(2\times 3)+(3\times 4)+\cdots+(n-1)n=\frac{(n-1)n(n+1)}{3}$$

(i)



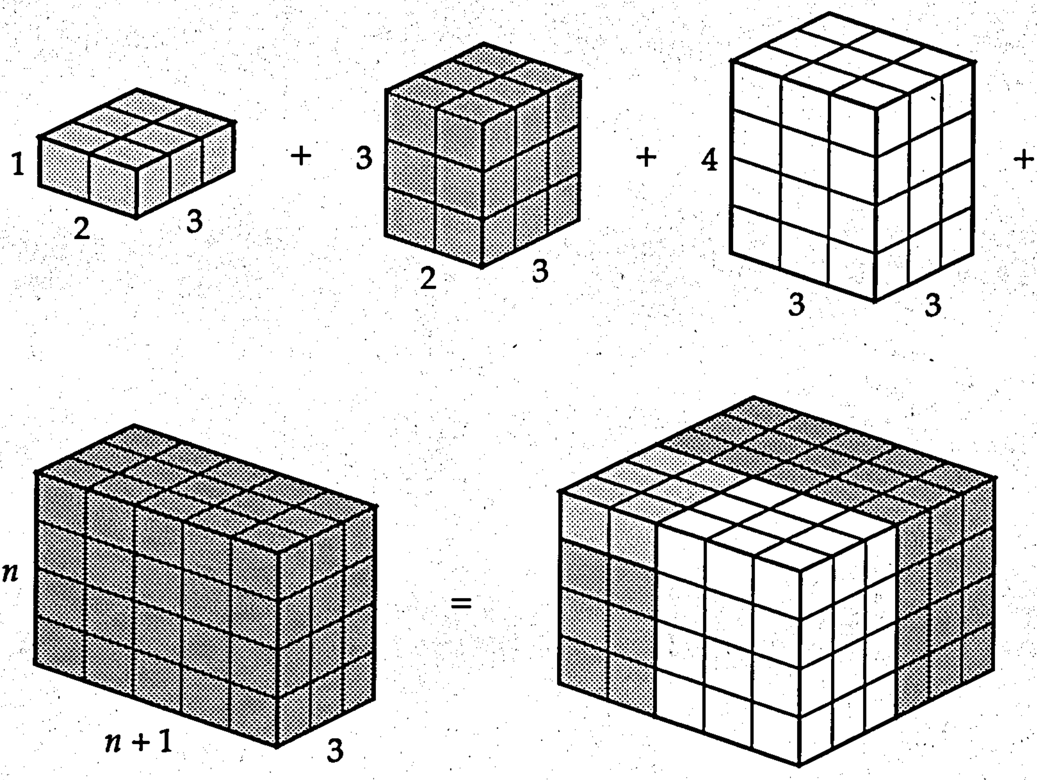
(ii)



—T. C. Wu

Sums of Oblong Numbers II

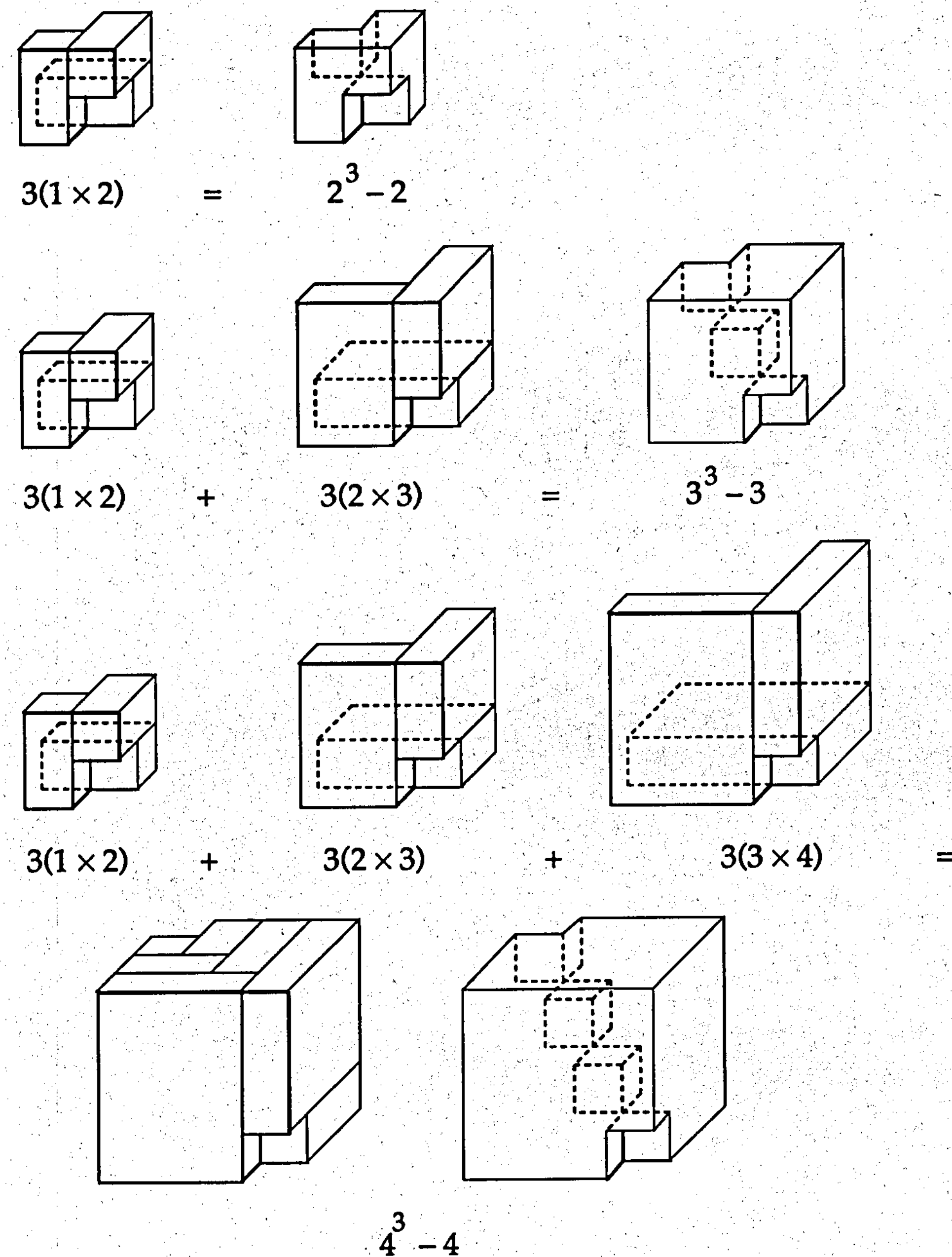
$$3(1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + n(n+1)) = n(n+1)(n+2)$$



—Sidney H. Kung

Sums of Oblong Numbers III

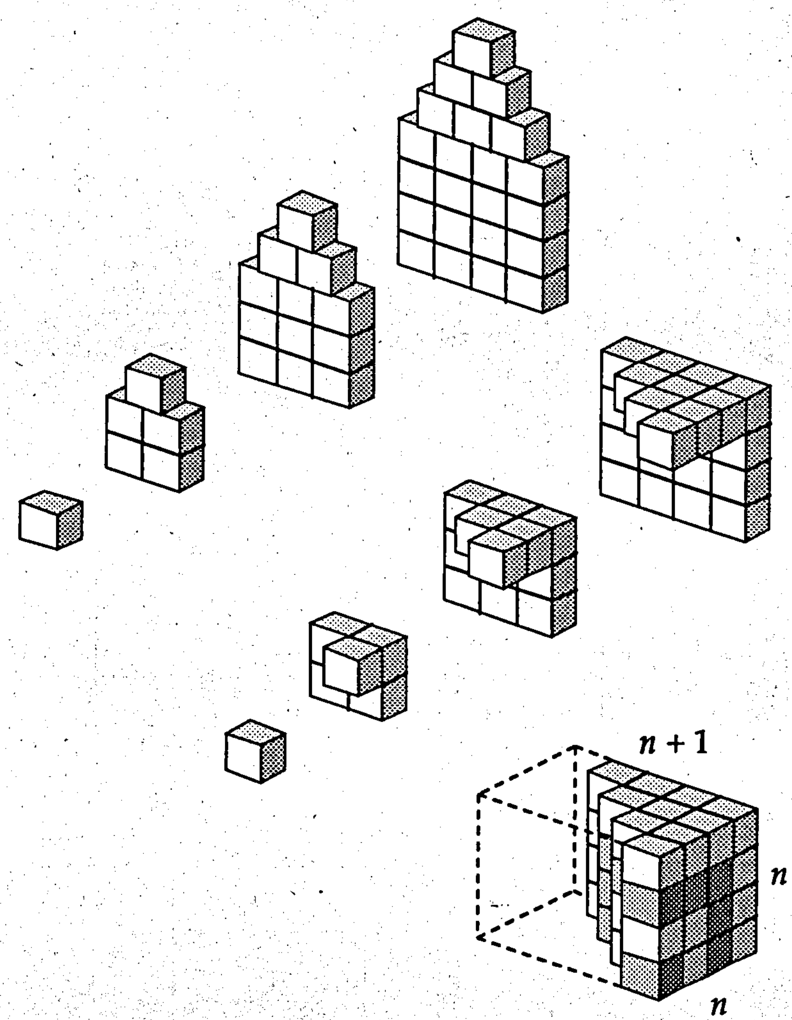
$$(1 \times 2) + (2 \times 3) + \cdots + (n-1) \times n = \frac{1}{3}[n^3 - n]$$



—Ali R. Amir-Moéz

Sums of Pentagonal Numbers

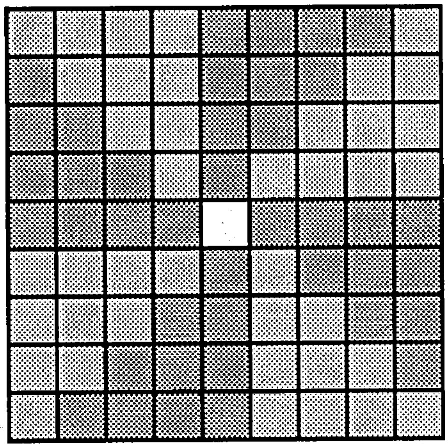
$$\frac{1\cdot 2}{2}+\frac{2\cdot 5}{2}+\frac{3\cdot 8}{2}+\cdots+\frac{n(3n-1)}{2}=\frac{n^2(n+1)}{2}$$



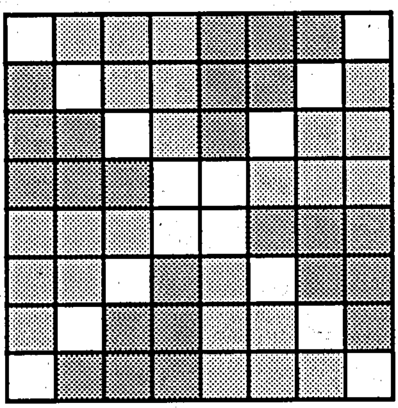
—William A. Miller

On Squares of Positive Integers

$$T_n = 1 + 2 + \cdots + n \quad \Rightarrow$$



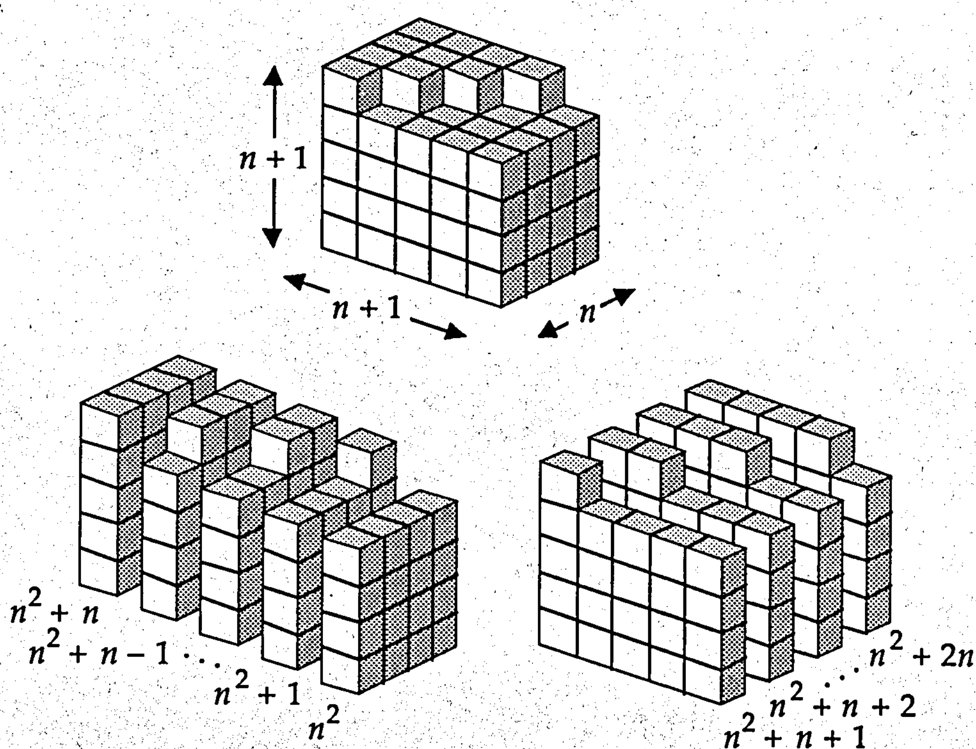
$$(2n+1)^2 = 8T_n + 1$$



$$(2n)^2 = 8T_{n-1} + 4n$$

—Edwin G. Landauer

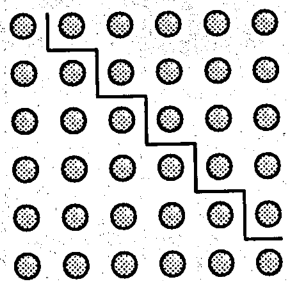
Consecutive Sums of Consecutive Integers



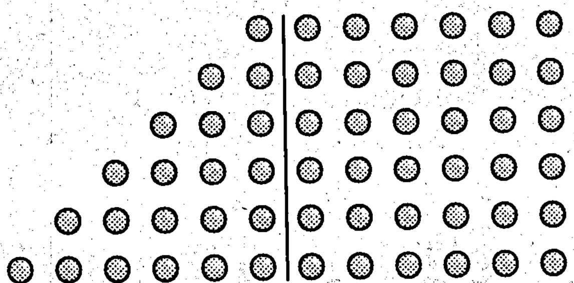
$$1 + 2 = 3$$
$$4 + 5 + 6 = 7 + 8$$
$$9 + 10 + 11 + 12 = 13 + 14 + 15$$
$$16 + 17 + 18 + 19 + 20 = 21 + 22 + 23 + 24$$
$$\vdots$$
$$n^2 + (n^2 + 1) + \cdots + (n^2 + n) = (n^2 + n + 1) + \cdots + (n^2 + 2n)$$

—RBN

Count the Dots



$$\sum_{k=1}^{n} k + \sum_{k=1}^{n-1} k = n^2$$

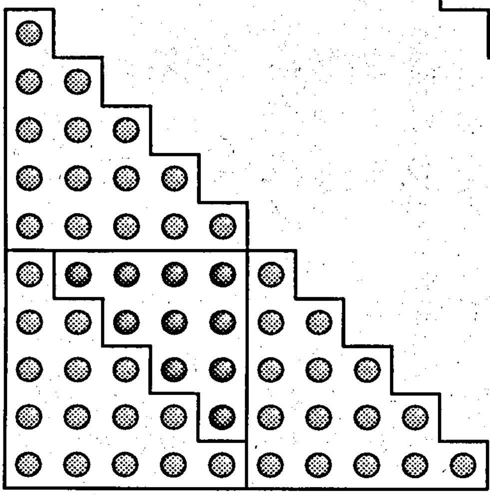


$$\sum_{k=1}^{n} k + n^2 = \sum_{k=n+1}^{2n} k$$

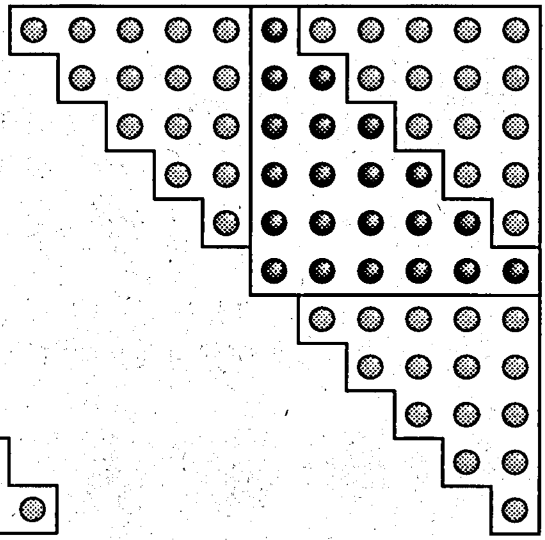
—Warren Page

Identities for Triangular Numbers

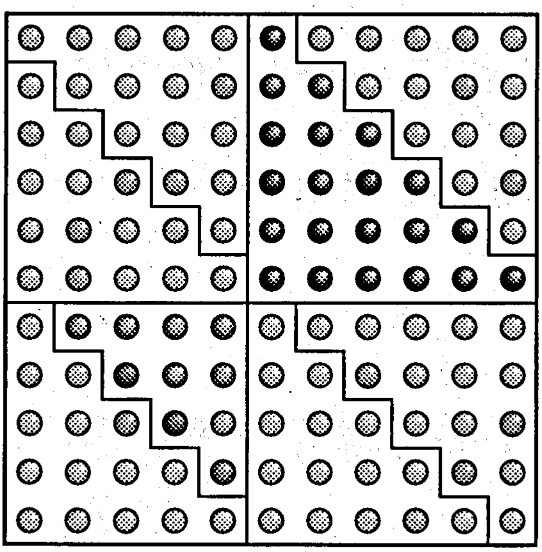
$$T_n = 1 + 2 + \cdots + n \quad \Rightarrow$$



$$3T_n + T_{n-1} = T_{2n}$$

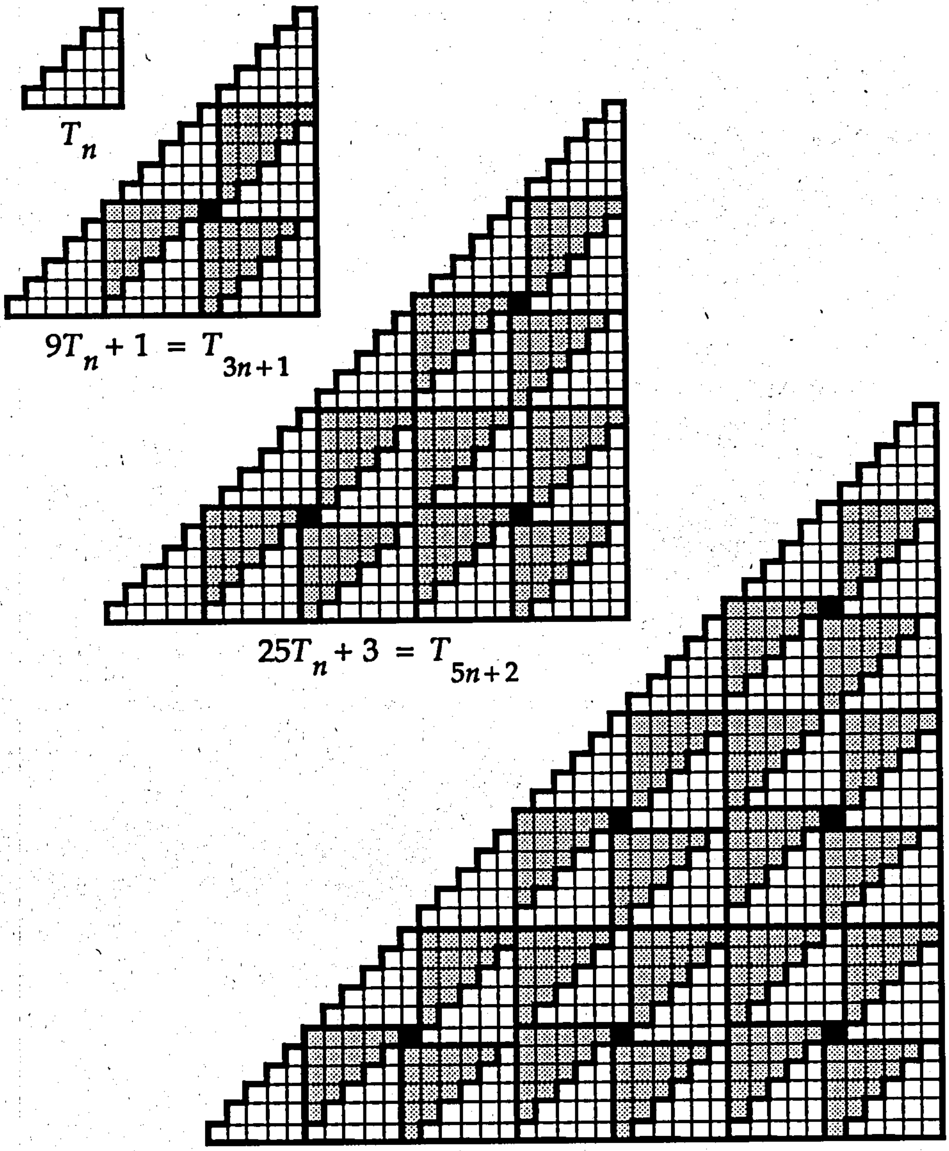


$$3T_n + T_{n+1} = T_{2n+1}$$



$$T_{n-1} + 6T_n + T_{n+1} = (2n+1)^2$$

A Triangular Identity



T_n

$9T_n + 1 = T_{3n+1}$

$25T_n + 3 = T_{5n+2}$

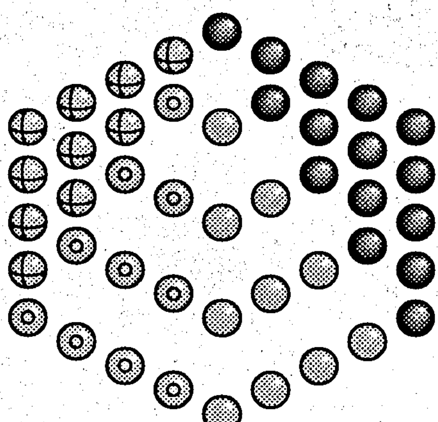
$49T_n + 6 = T_{7n+3}$

$$T_n = 1 + 2 + \cdots + n \Rightarrow (2k+1)^2 T_n + T_k = T_{(2k+1)n+k}$$

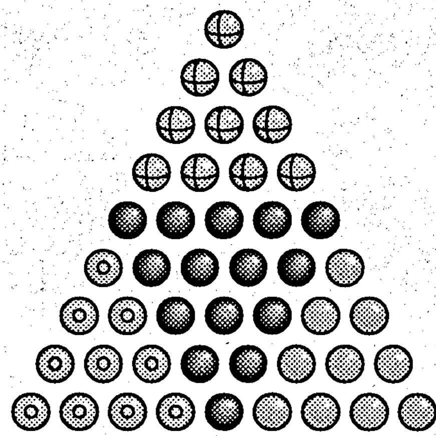
—RBN

Every Hexagonal Number is a Triangular Number

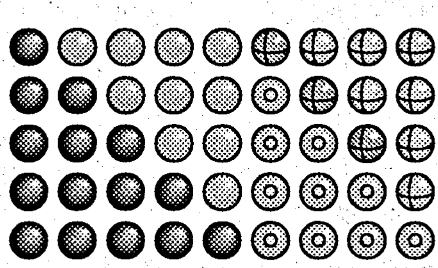
$$\left.\begin{aligned} H_n &= 1+5+\cdots+(4n-3) \\ T_n &= 1+2+\cdots+n \end{aligned}\right\} \Rightarrow H_n = 3T_{n-1} + T_n = T_{2n-1} = n(2n-1)$$



H_5

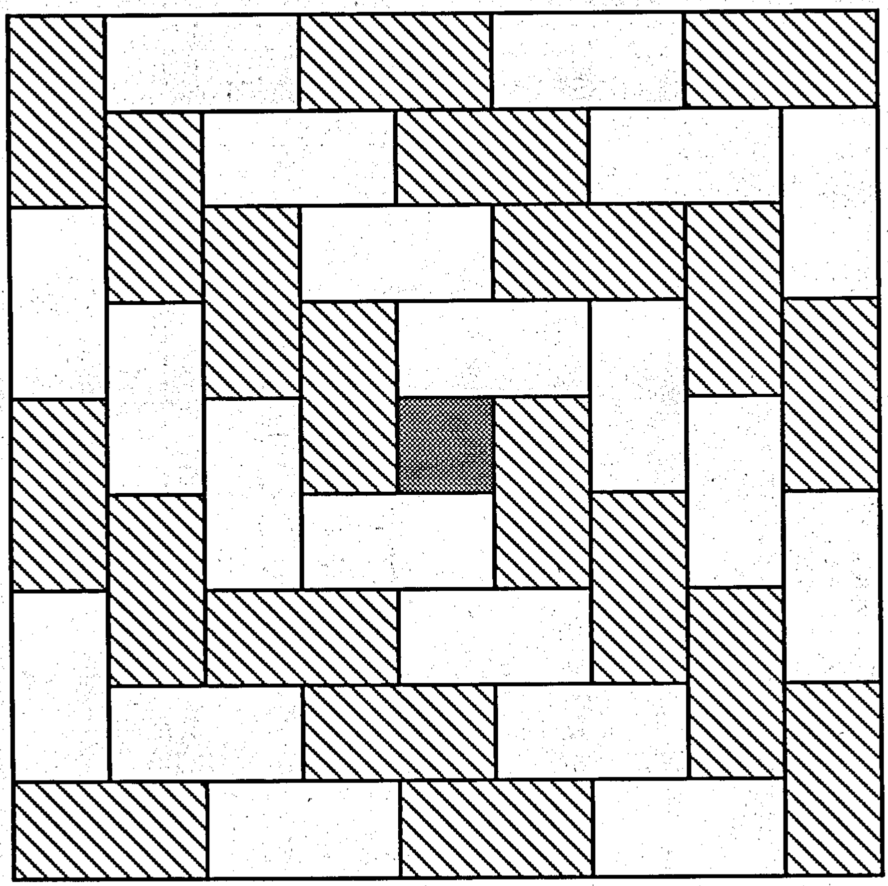


T_9



5·9

One Domino = Two Squares: Concentric Squares

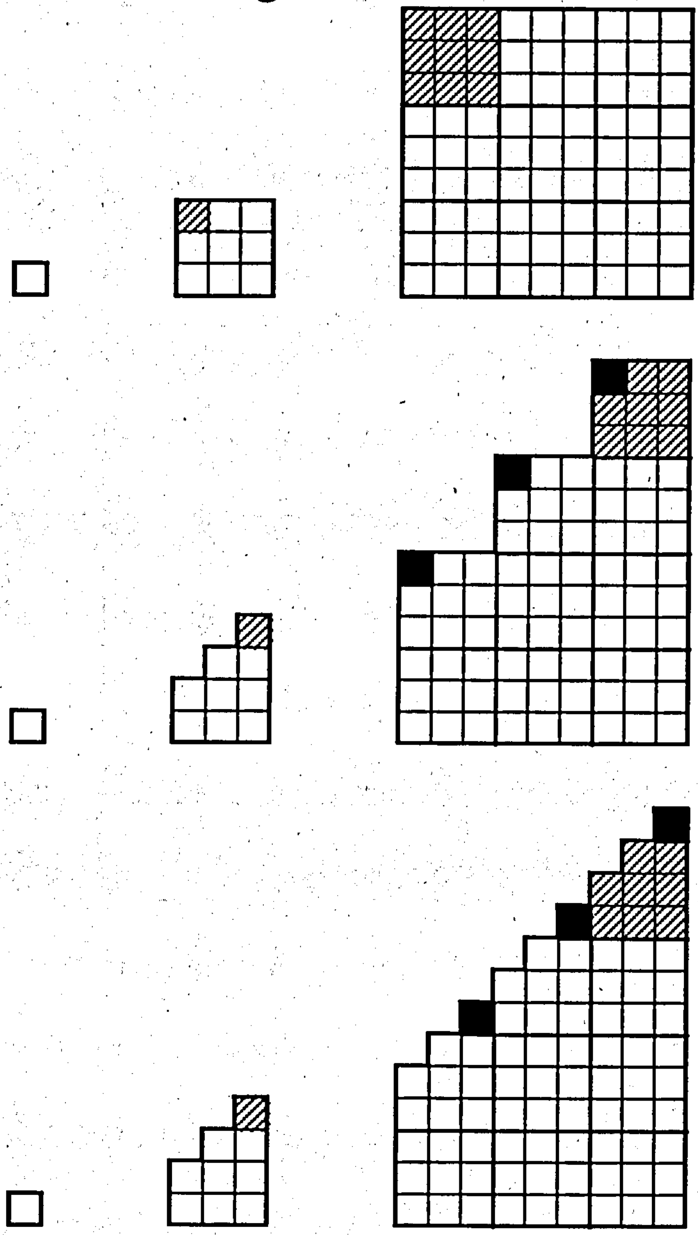


$$1 + 4 \cdot 2 + 8 \cdot 2 + 12 \cdot 2 + 16 \cdot 2 = 9^2$$

$$1 + 2 \sum_{k=1}^{n} 4k = (2n+1)^2$$

—Shirley A. Wakin

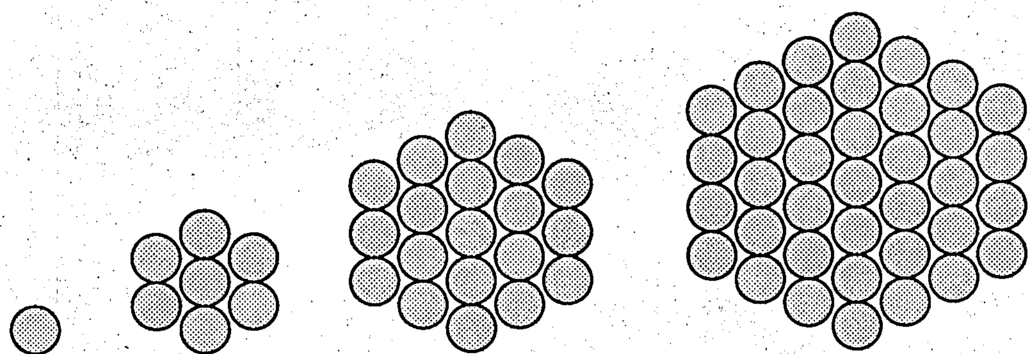
Sums of Consecutive Powers of Nine are Sums of Consecutive Integers



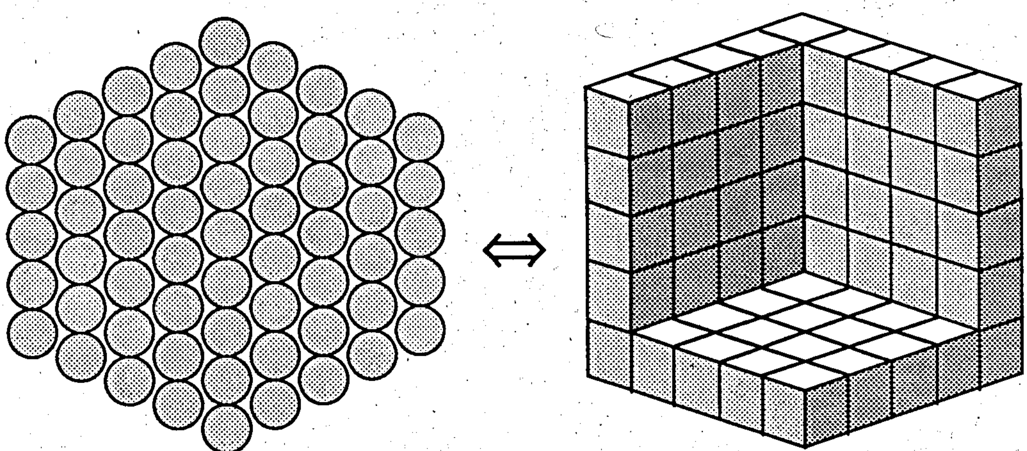
$$1 + 9 + \cdots + 9^n = 1 + 2 + 3 + \cdots + (1 + 3 + \cdots + 3^n)$$

—RBN

Sums of Hex Numbers Are Cubes



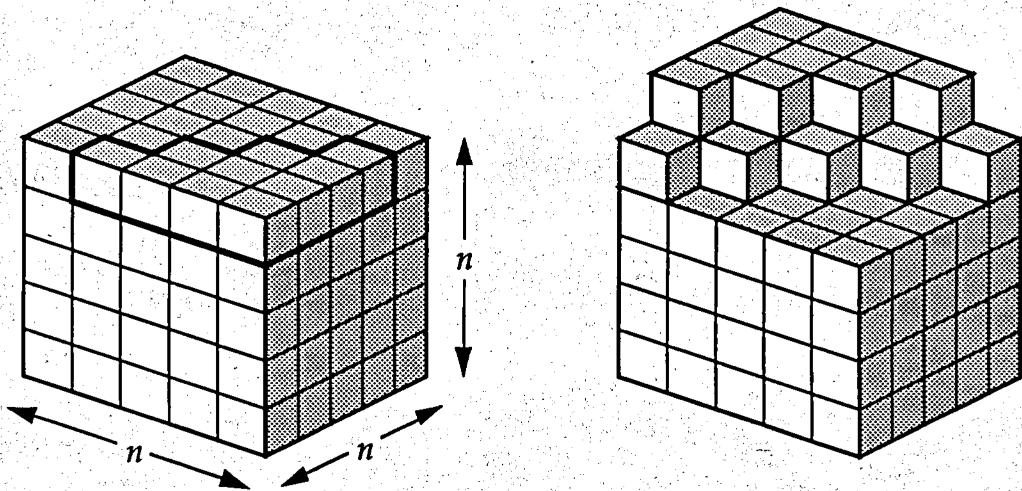
$h_1 = 1 \quad h_2 = 7 \quad h_3 = 19 \quad h_4 = 37$

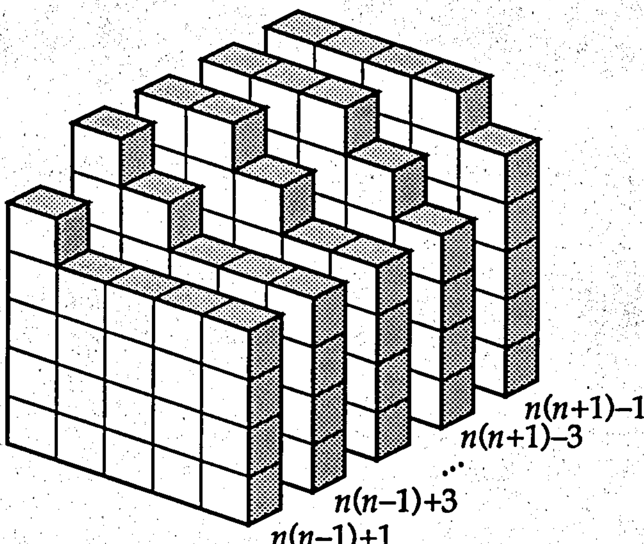


$$h_n = n^3 - (n-1)^3$$

$$\therefore h_1 + h_2 + \cdots + h_n = n^3.$$

Every Cube is the Sum of Consecutive Odd Numbers





$1^3 = 1$

$2^3 = 3 + 5$

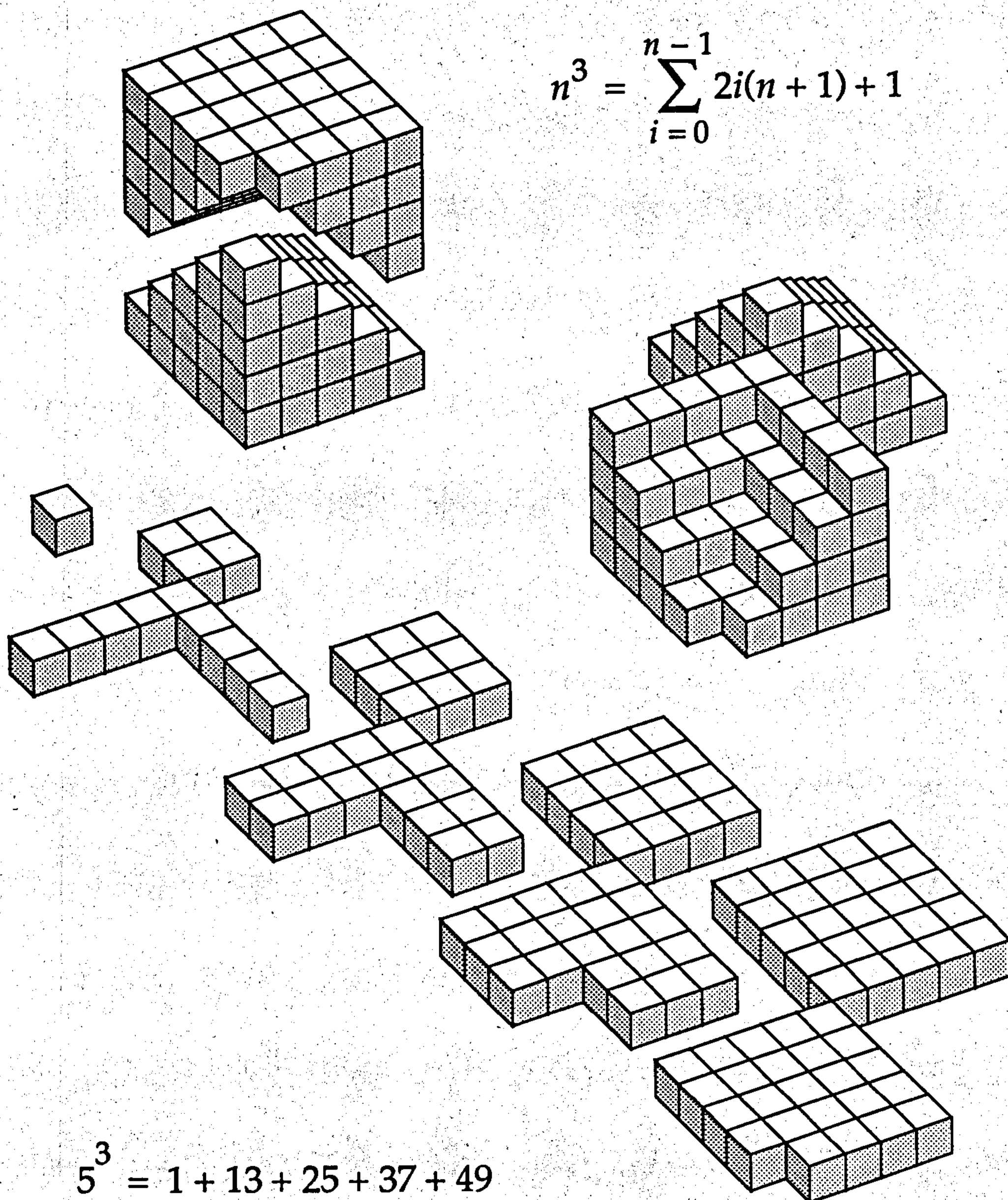
$3^3 = 7 + 9 + 11$

$\vdots$

$n^3 = [n(n-1)+1] + \cdots + [n(n+1)-1]$

—RBN

The Cube as an Arithmetic Sum



$$n^3 = \sum_{i=0}^{n-1} 2i(n+1)+1$$

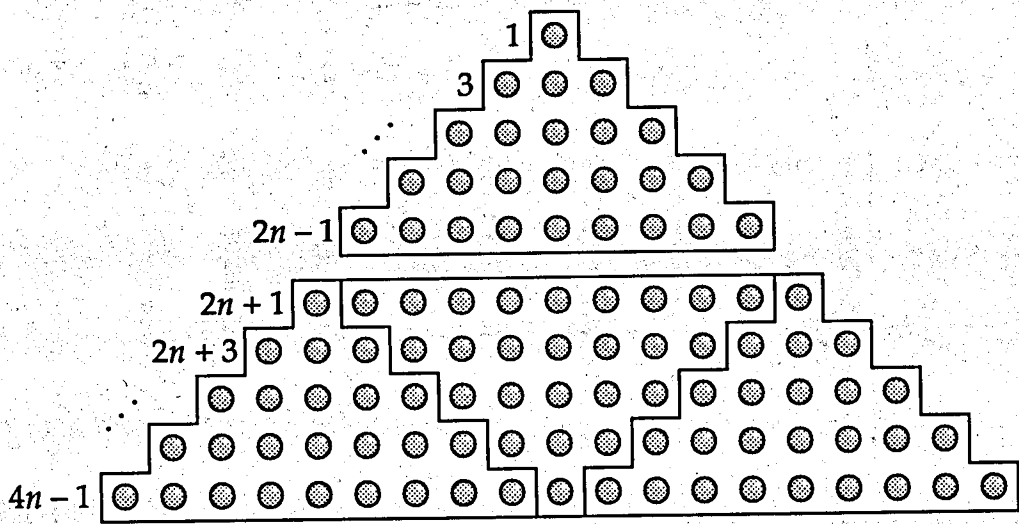
$$5^3 = 1 + 13 + 25 + 37 + 49$$

—Robert Bronson and
Christopher Brueningsen

Sequences & Series

On a Property of the Sequence of Odd Integers (Galileo, 1615)

$$\frac{1}{3} = \frac{1+3}{5+7} = \frac{1+3+5}{7+9+11} = \cdots .$$



$$\frac{1+3+\ldots+(2n-1)}{(2n+1)+(2n+3)+\ldots+(4n-1)} = \frac{1}{3}$$

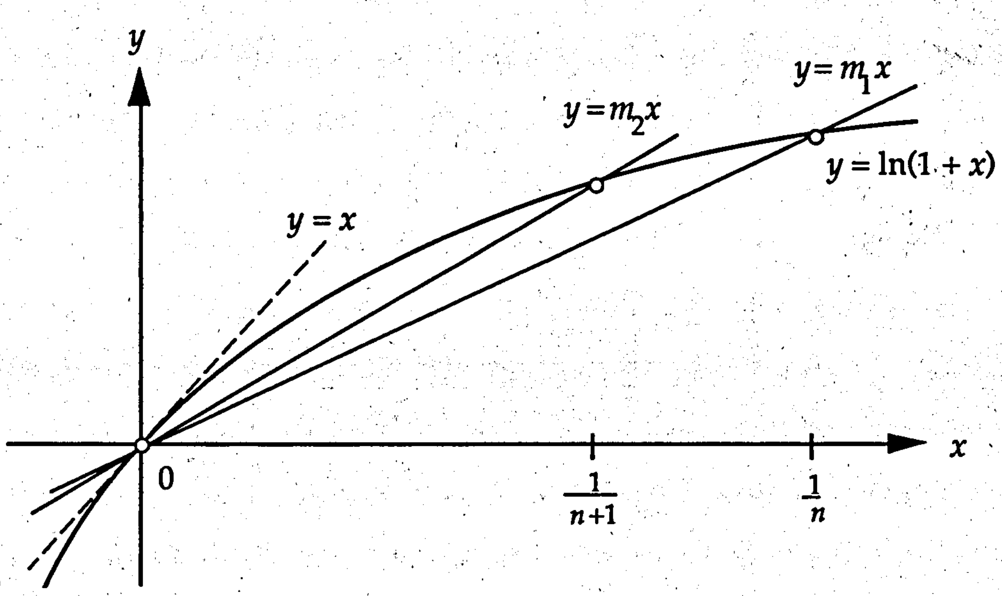
REFERENCE

S. Drake, *Galileo Studies*, The University of Michigan Press, Ann Arbor, 1970, pp. 218-219.

—RBN

A Monotone Sequence Bounded by e

$$\forall n \geq 1,\ \left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1} < e.$$



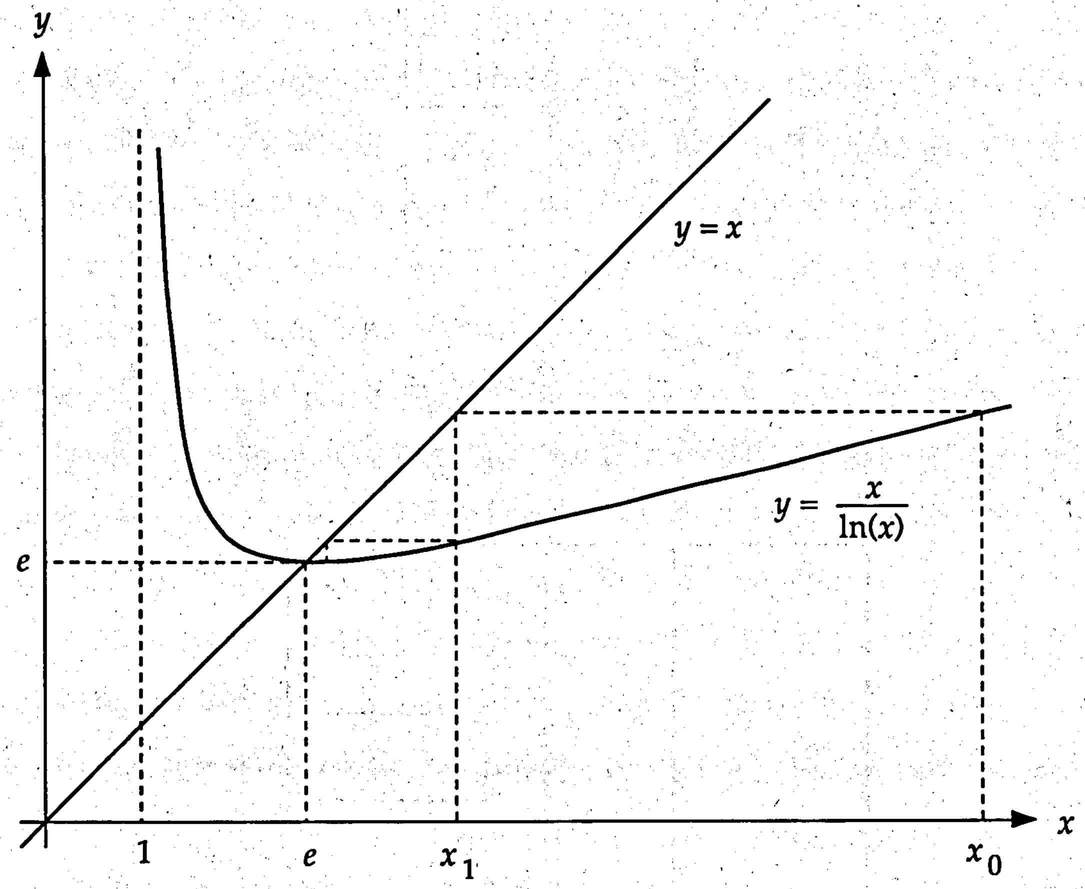
$$n \geq 1 \Rightarrow m_1 < m_2 < 1$$

$$\Rightarrow \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} < \frac{\ln\left(1+\frac{1}{n+1}\right)}{\frac{1}{n+1}} < 1$$

$$\Rightarrow \left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1} < e$$

—RBN

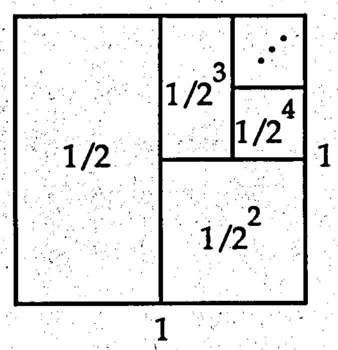
A Recusively Defined Sequence for e



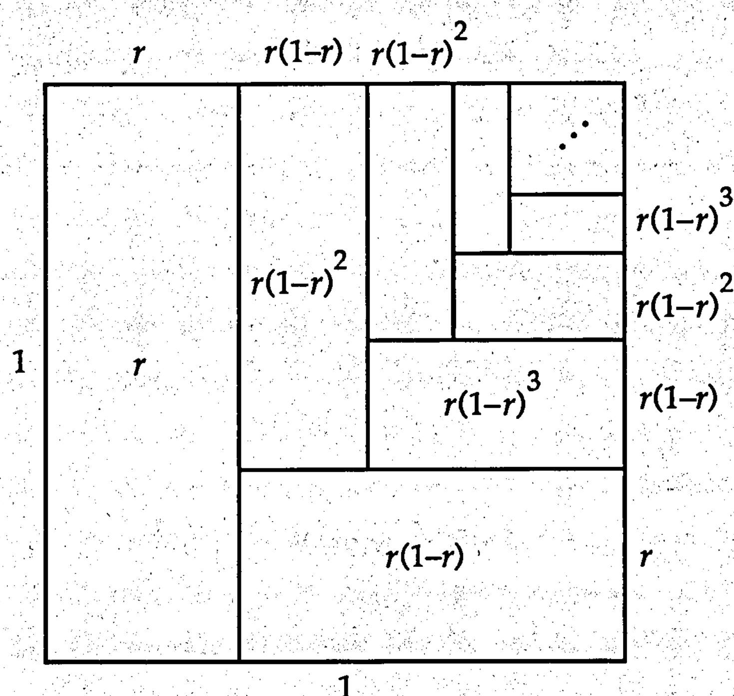
$$x_0 > 1 \;\&\; x_{n+1} = \frac{x_n}{\ln(x_n)} \Rightarrow \lim x_n = e$$

—Thomas P. Dence

Geometric Sums



$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$$

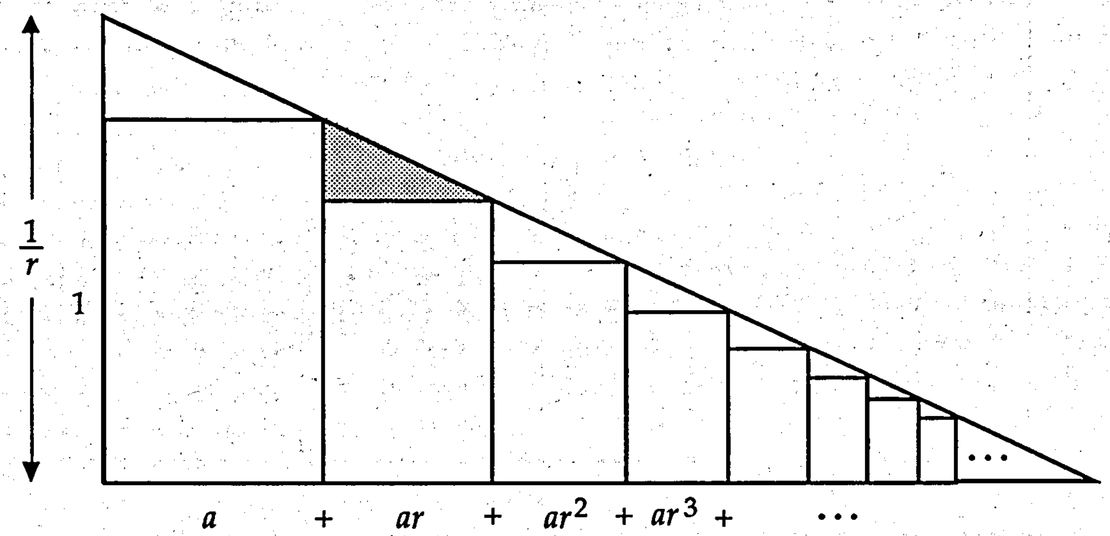


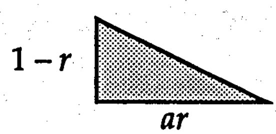
$$r + r(1-r) + r(1-r)^2 + \cdots = 1$$

—Warren Page

Geometric Series I

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

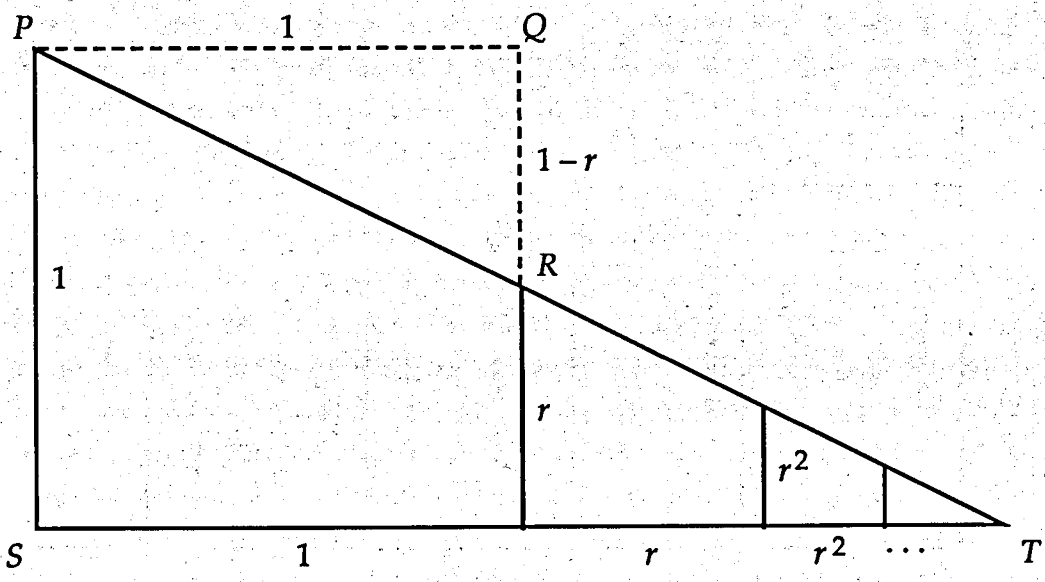




$$\frac{a + ar + ar^2 + ar^3 + \cdots}{1/r} = \frac{ar}{1-r}$$

—J. H. Webb

Geometric Series II

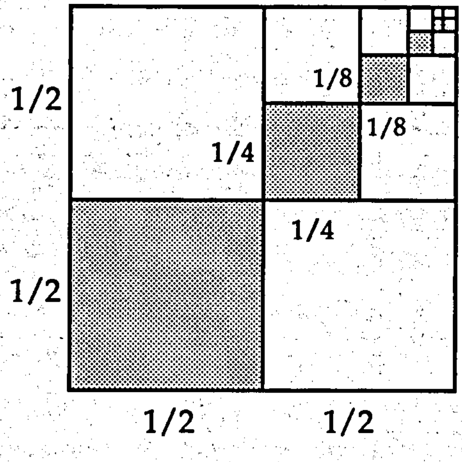


$$\Delta PQR \approx \Delta TSP$$

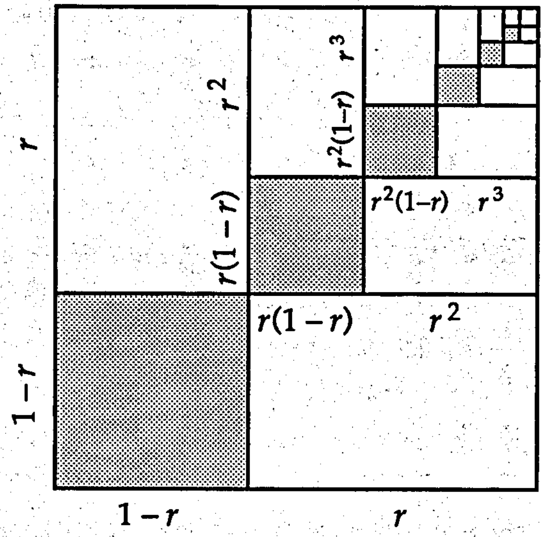
$$\therefore 1 + r + r^2 + \ldots = \frac{1}{1 - r}.$$

—Benjamin G. Klein and Irl C. Bivens

Geometric Series III



$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \cdots = \frac{1}{3}$$



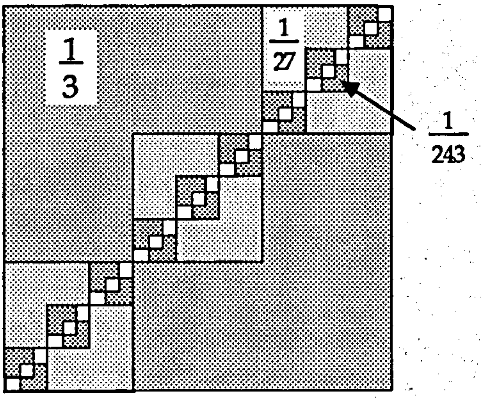
$$(1-r)^2 + r^2(1-r)^2 + r^4(1-r)^2 + \ldots = \frac{(1-r)^2}{(1-r)^2 + 2r(1-r)} = \frac{1-r}{1+r}$$

$$1 + r^2 + r^4 + \ldots = \frac{1}{1-r^2}$$

$$a + ar + ar^2 + \ldots = \frac{a}{1-r}$$

—Sunday A. Ajose

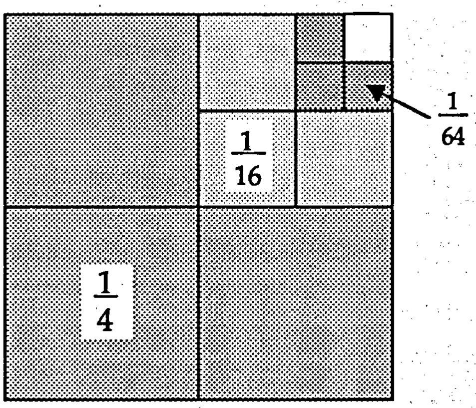
Geometric Series IV



$$2\left(\frac{1}{3}+3\cdot\frac{1}{27}+9\cdot\frac{1}{243}+\cdots\right)=1$$

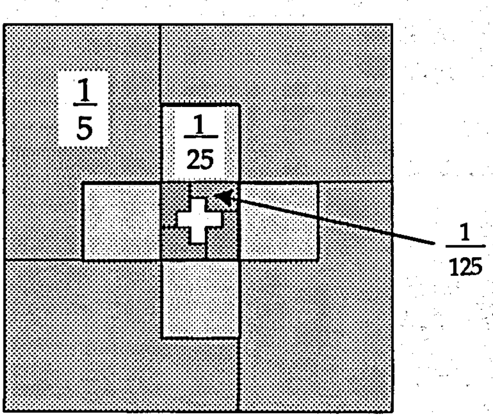
$$2\sum_{n=1}^{\infty}\frac{1}{3^n}=1$$

$$\sum_{n=1}^{\infty}\frac{1}{3^n}=\frac{1}{2}$$



$$3\sum_{n=1}^{\infty}\frac{1}{4^n}=1$$

$$\sum_{n=1}^{\infty}\frac{1}{4^n}=\frac{1}{3}$$



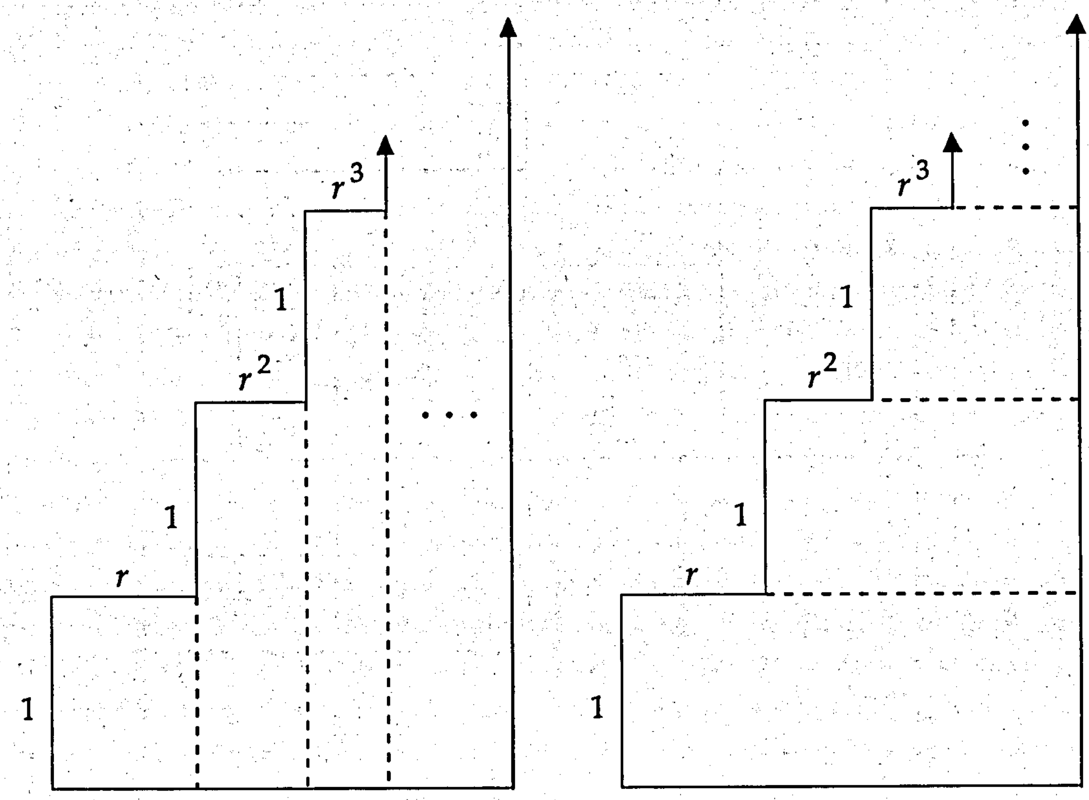
$$4\sum_{n=1}^{\infty}\frac{1}{5^n}=1$$

$$\sum_{n=1}^{\infty}\frac{1}{5^n}=\frac{1}{4}$$

—Elizabeth M. Markham

Gabriel's Staircase

$$\sum_{k=1}^{\infty} kr^k = \frac{r}{(1-r)^2} \text{ for } 0 < r < 1$$

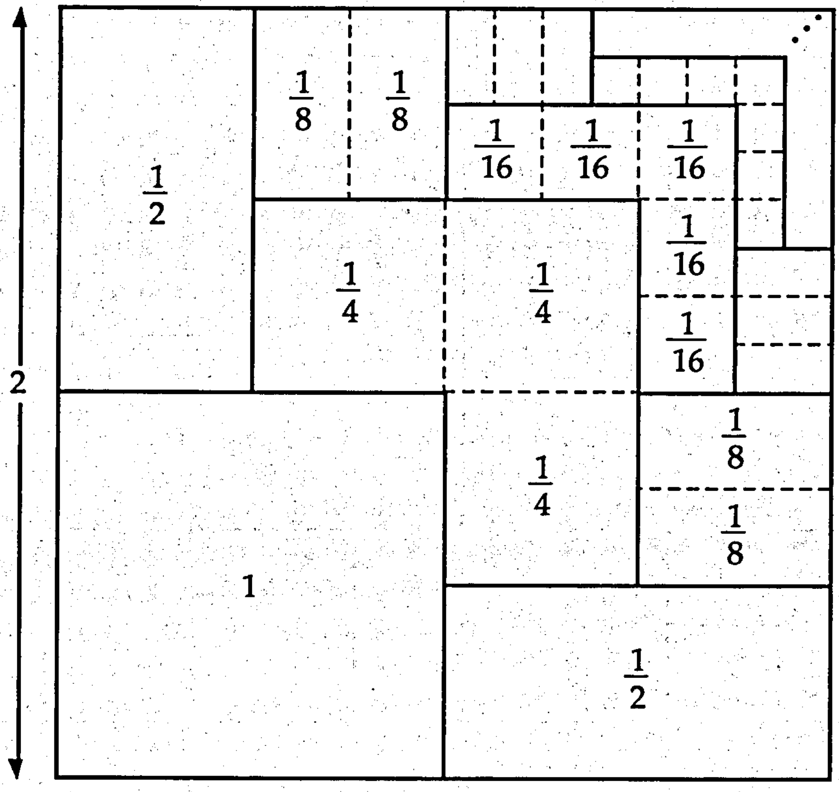


$$\sum_{k=1}^{\infty} kr^k = \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} r^i = \frac{r}{(1-r)^2}$$

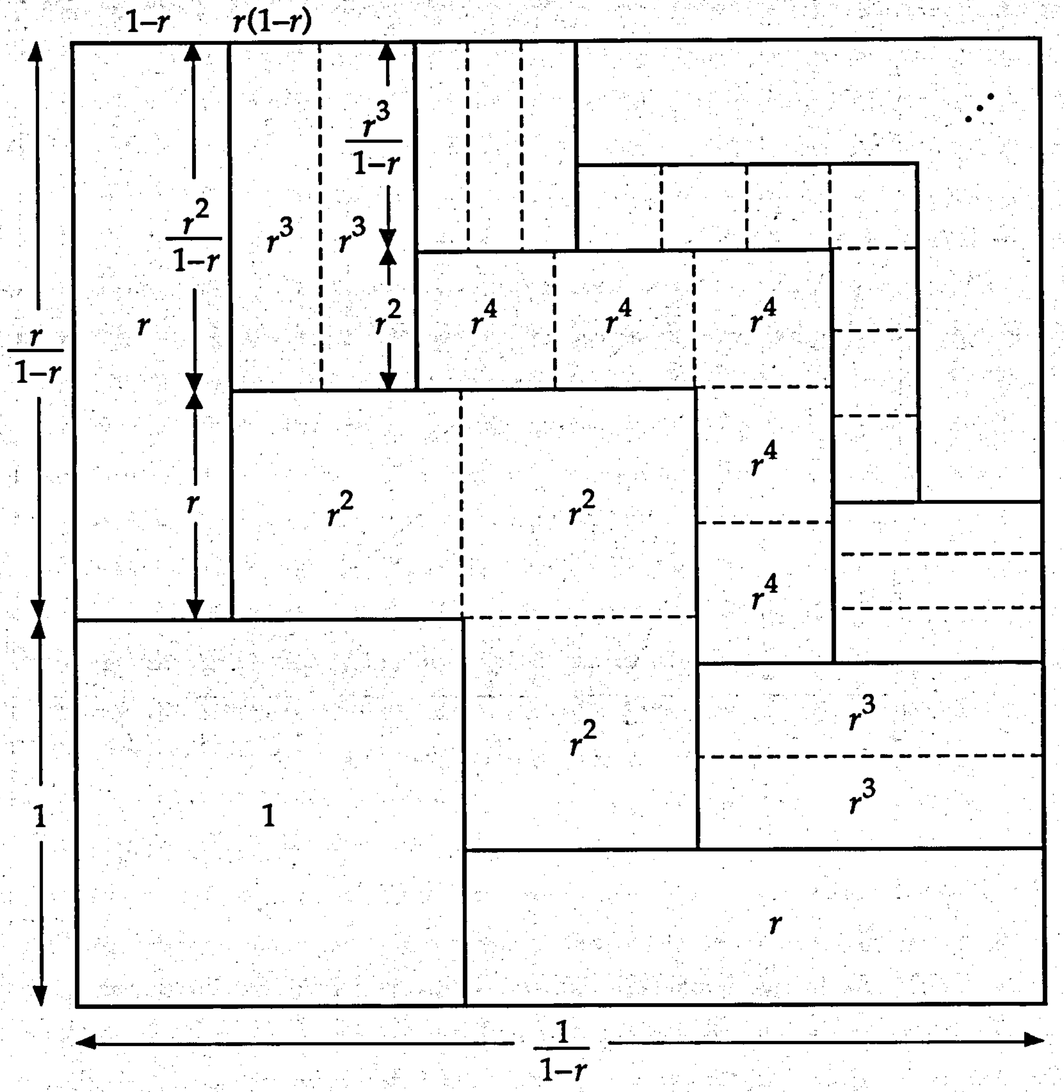
—Stuart G. Swain

Differentiated Geometric Series

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) + \cdots = 4$$

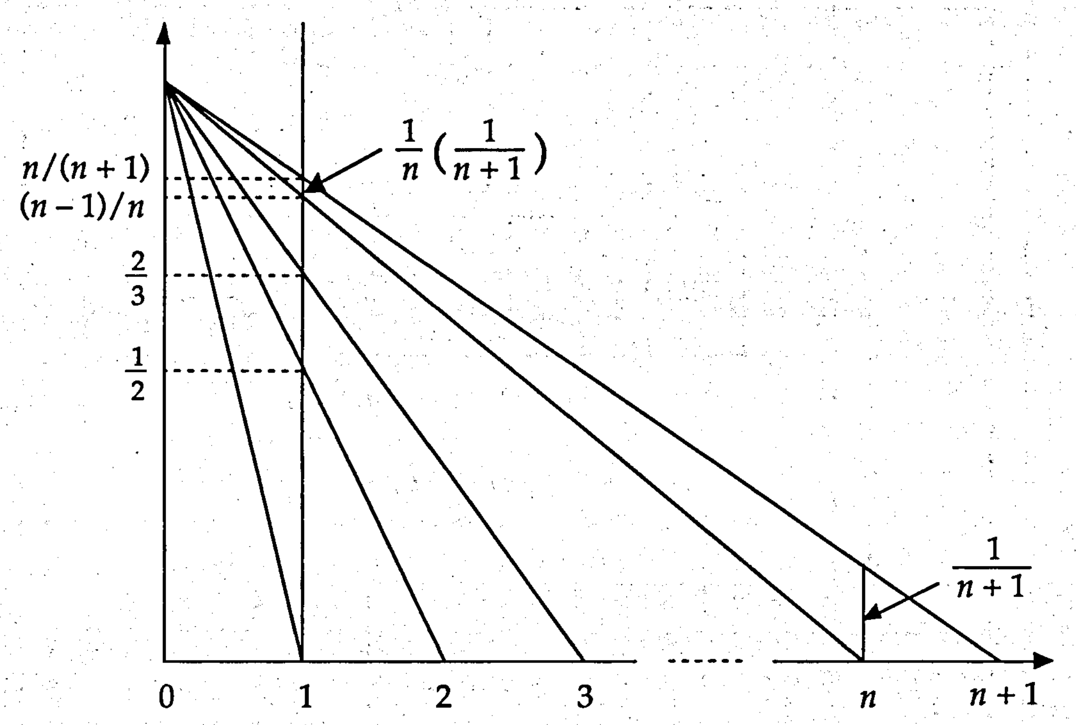


$$1 + 2r + 3r^2 + 4r^3 + \ldots = \left(\frac{1}{1-r}\right)^2,\ 0 \le r < 1$$



—RBN

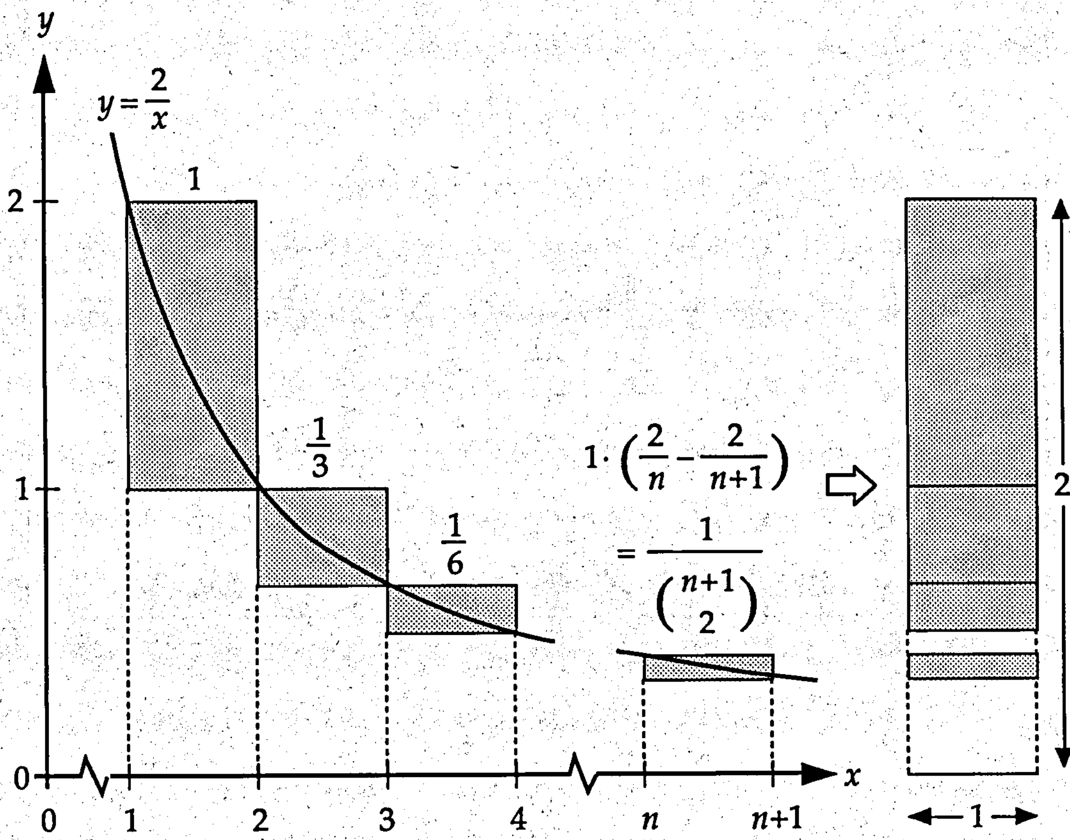
$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$



—Roman W. Wong

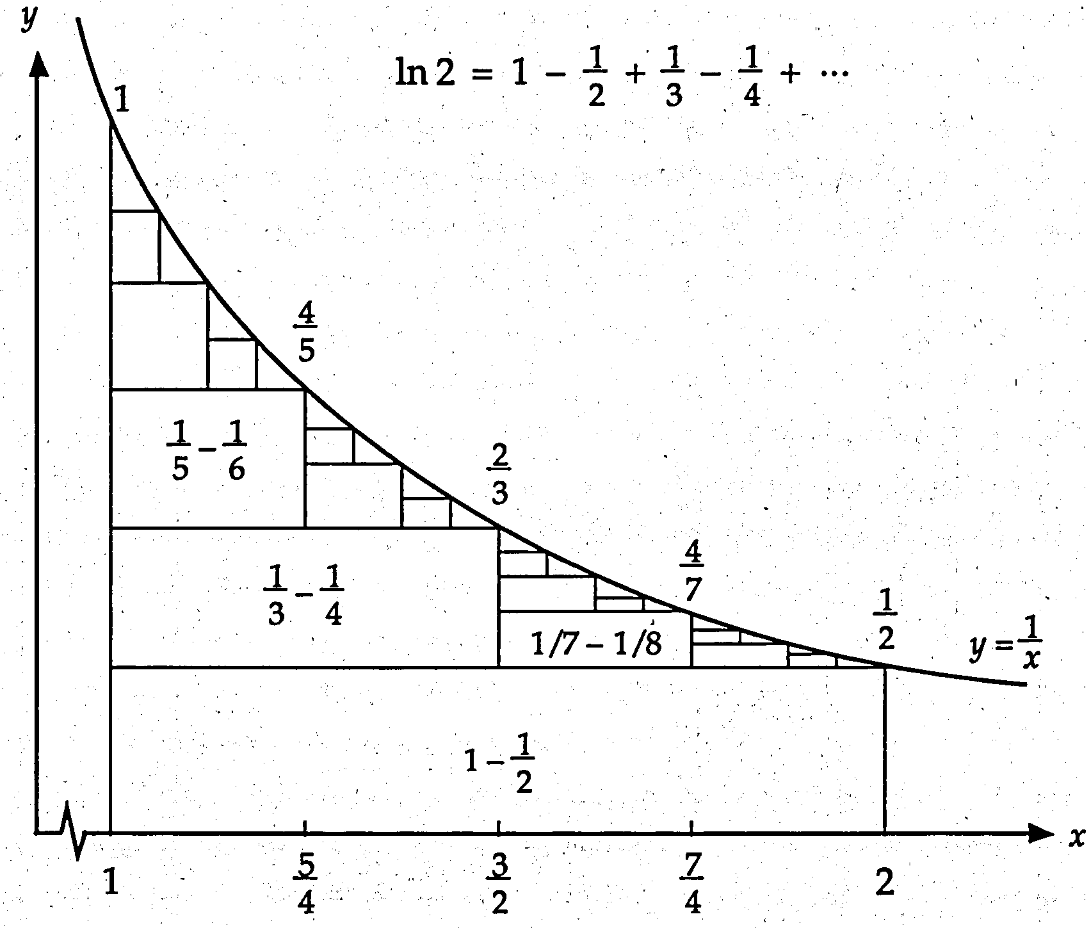
The Series of Reciprocals of Triangular Numbers

$$\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\cdots+\frac{1}{\binom{n+1}{2}}+\cdots=2$$



—RBN

The Alternating Harmonic Series



$$\frac{1}{2}\left(\frac{2}{3}-\frac{2}{4}\right)=\frac{1}{3}-\frac{1}{4};$$

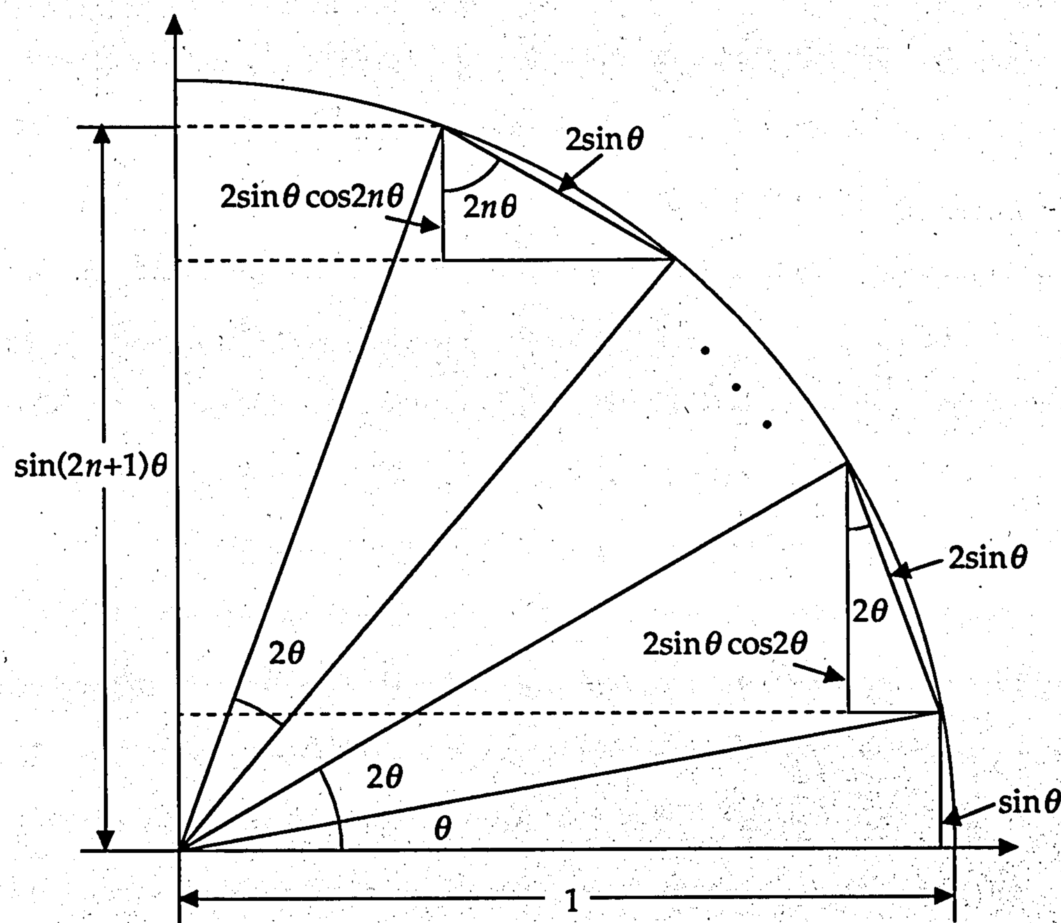
$$\frac{1}{4}\left(\frac{4}{5}-\frac{4}{6}\right)=\frac{1}{5}-\frac{1}{6}, \qquad \frac{1}{4}\left(\frac{4}{7}-\frac{4}{8}\right)=\frac{1}{7}-\frac{1}{8};$$

$$\frac{1}{2^n}\left(\frac{2^n}{2^n+2k-1}-\frac{2^n}{2^n+2k}\right)=\frac{1}{2^n+2k-1}-\frac{1}{2^n+2k}, \quad k=1,2,\cdots,2^{n-1}; \quad n=1,2,\cdots.$$

$$\ln 2=\int_1^2\frac{dx}{x}=\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots.$$

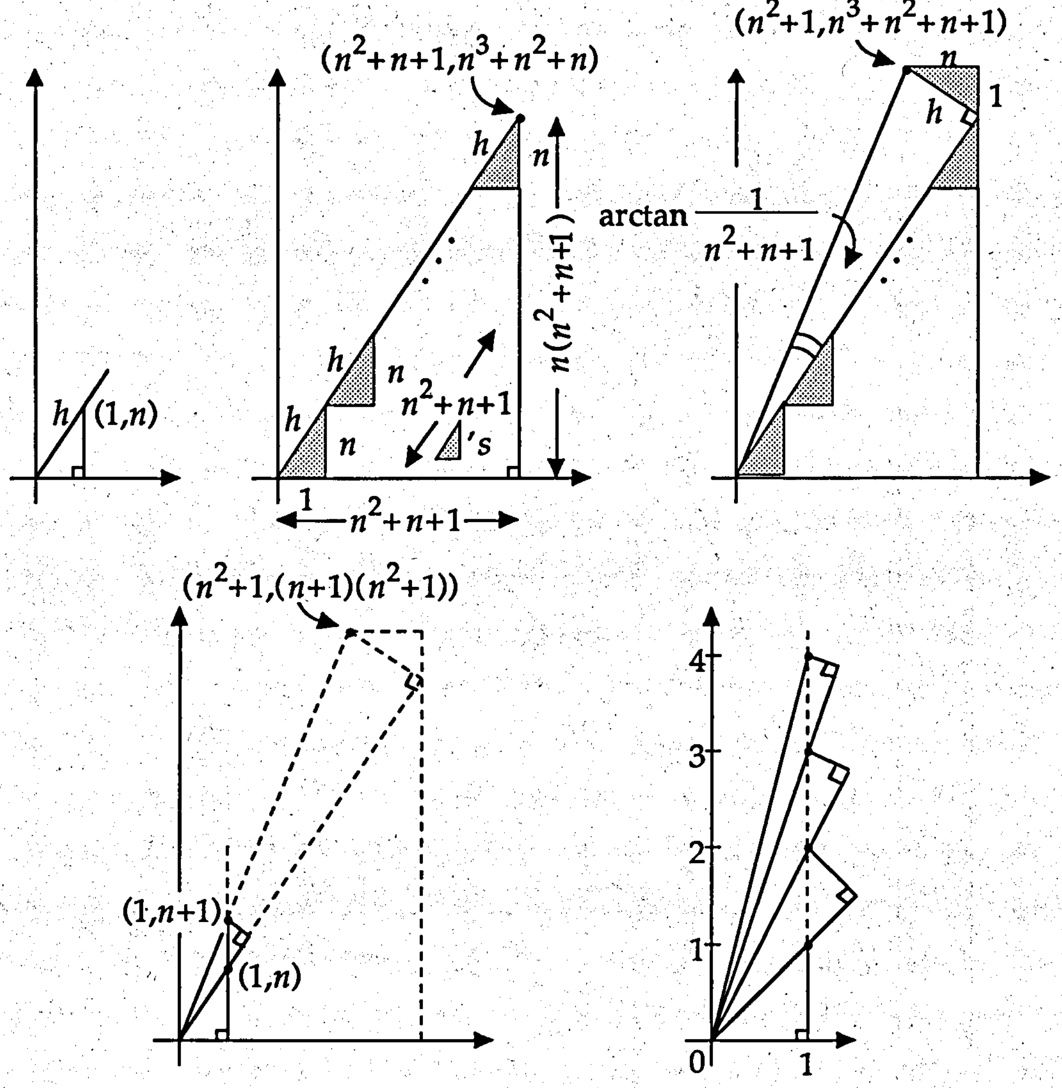
—Mark Finkelstein

$$\sin(2n+1)\theta = \sin\theta + 2\sin\theta \sum_{k=1}^{n} \cos 2k\theta$$



—J. Chris Fisher and E. L. Koh

An Arctangent Identity and Series



$$\arctan n + \arctan \frac{1}{n^2+n+1} = \arctan(n+1)$$

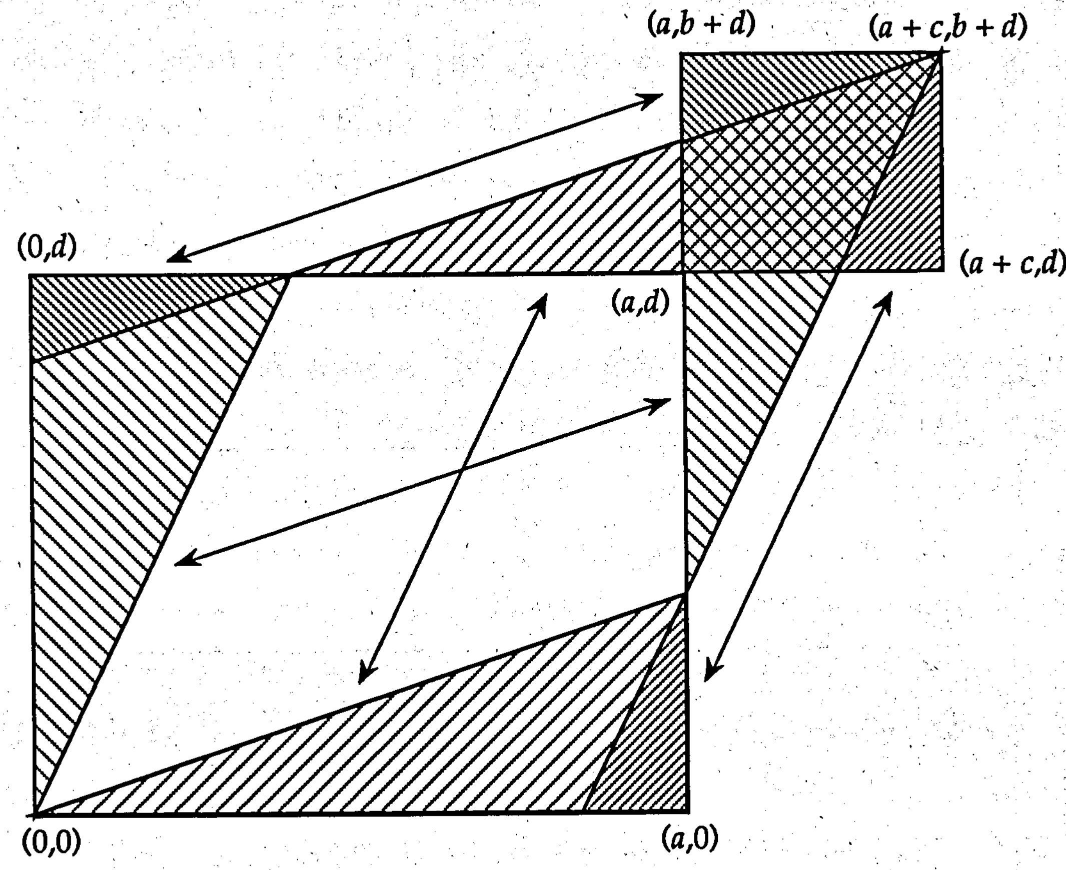
$$\arctan \frac{1}{n^2+n+1} = \arctan(n+1) - \arctan n$$

$$\sum_{n=0}^{\infty} \arctan \frac{1}{n^2+n+1} = \lim_{N\to\infty} \arctan(N+1) = \frac{\pi}{2}$$

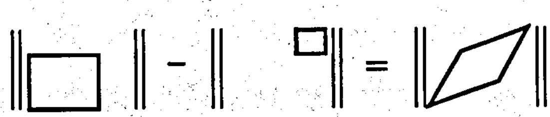
—RBN

Miscellaneous

A 2 × 2 Determinant is the Area of a Parallelogram

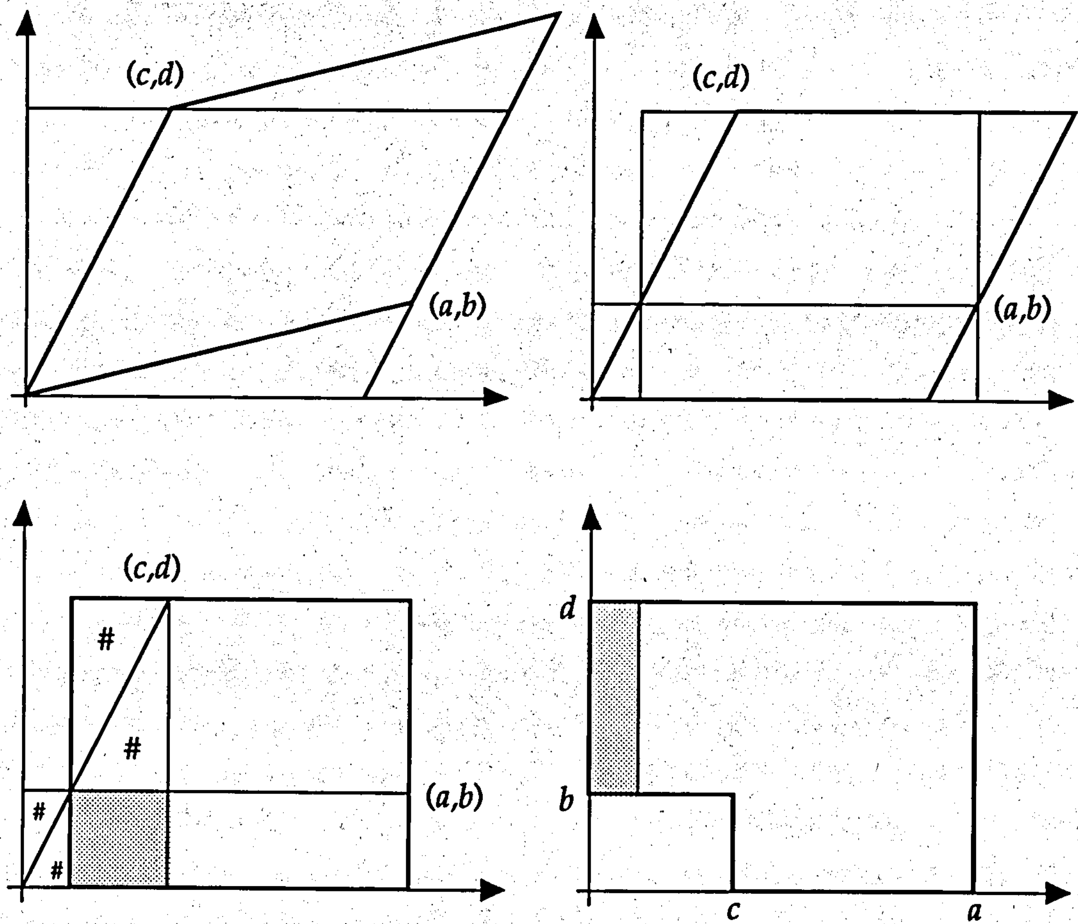


$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc =$$



—Solomon W. Golomb

Area of the Parallelogram Determined by Vectors (a,b) and $(c,d) = \pm \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \pm(ad - bc)$



—Yihnan David Gau

The Characteristic Polynomials of *AB* and *BA* are Equal

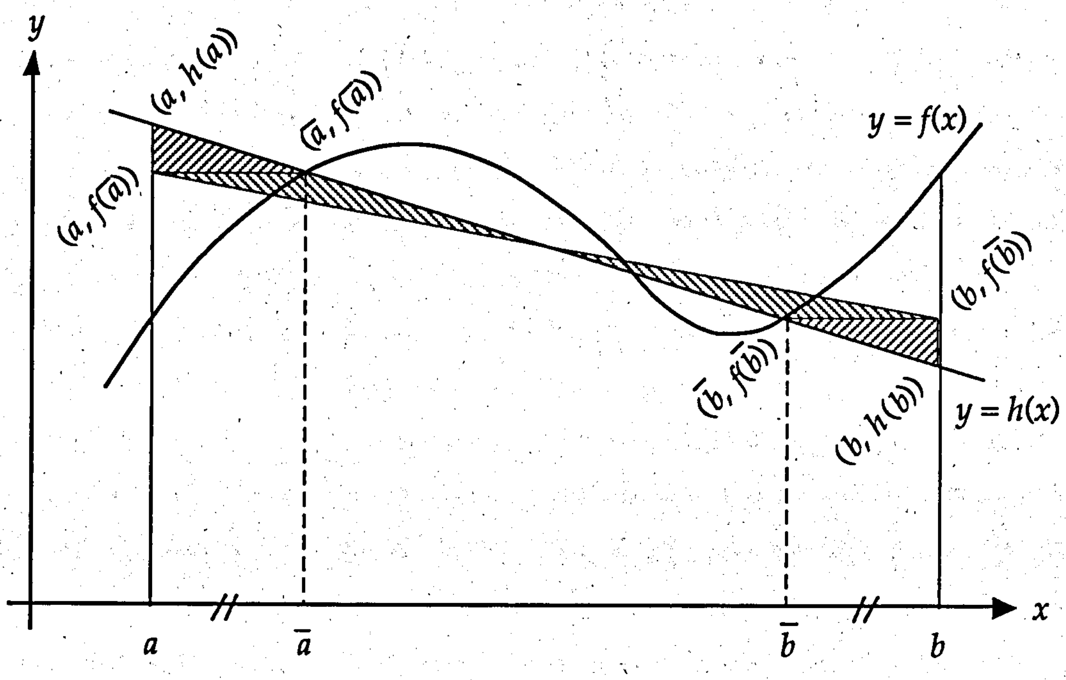
$$-\lambda^n|AB-\lambda I| = \left|\begin{pmatrix} A & AB-\lambda I \\ \lambda I & 0\end{pmatrix}\right| = \left|\begin{pmatrix} A & I \\ \lambda I & B\end{pmatrix}\begin{pmatrix} I & B \\ 0 & -\lambda I\end{pmatrix}\right| = \begin{vmatrix} A & I \\ \lambda I & B\end{vmatrix}(-\lambda)^n$$

$$-\lambda^n|BA-\lambda I| = \left|\begin{pmatrix} 0 & \lambda I \\ BA-\lambda I & \lambda B\end{pmatrix}\right| = \left|\begin{pmatrix} A & I \\ \lambda I & B\end{pmatrix}\begin{pmatrix} -I & 0 \\ A & \lambda I\end{pmatrix}\right| = \begin{vmatrix} A & I \\ \lambda I & B\end{vmatrix}(-\lambda)^n$$

—Sidney H. Kung

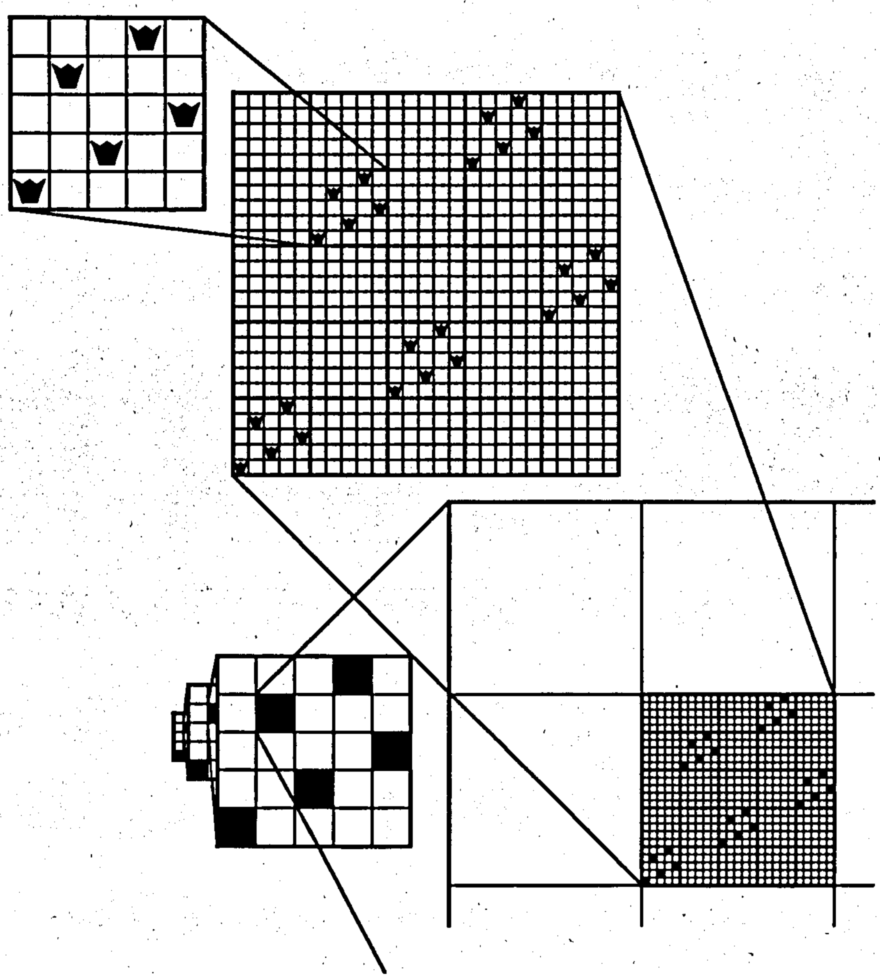
The Gaussian Quadrature as the Area of Either Trapezoid

$$\frac{1}{2}(b-a)(f(\overline{a})+f(\overline{b})) = \frac{1}{2}(b-a)(h(a)+h(b))$$



—Mike Akerman

Inductive Construction of an Infinite Chessboard with Maximal Placement of Nonattacking Queens



REFERENCES

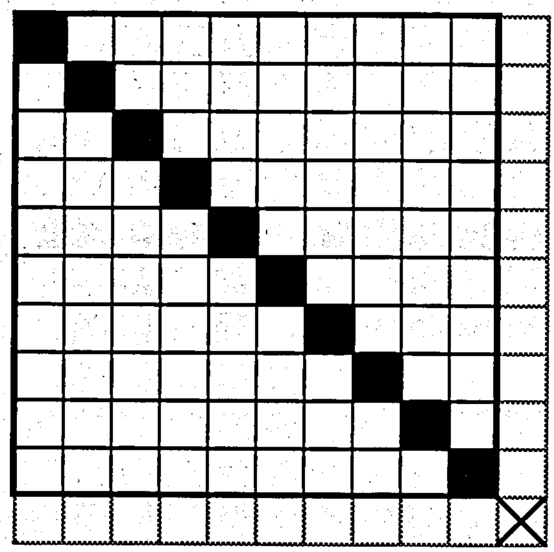
1. Dean S. Clark and Oved Shisha, Invulnerable Queens on an Infinite Chessboard, *Annals of the New York Academy of Sciences, The Third International Conference on Combinatorial Mathematics,* 1989, 133-139.
2. M. Kraitchik, *La Mathématique des Jeux ou Récréations Mathématiques,* Imprimerie Stevens Frères, Bruxelles, 1930, 349-353.

—Dean S. Clark and Oved Shisha

Combinatorial Identities

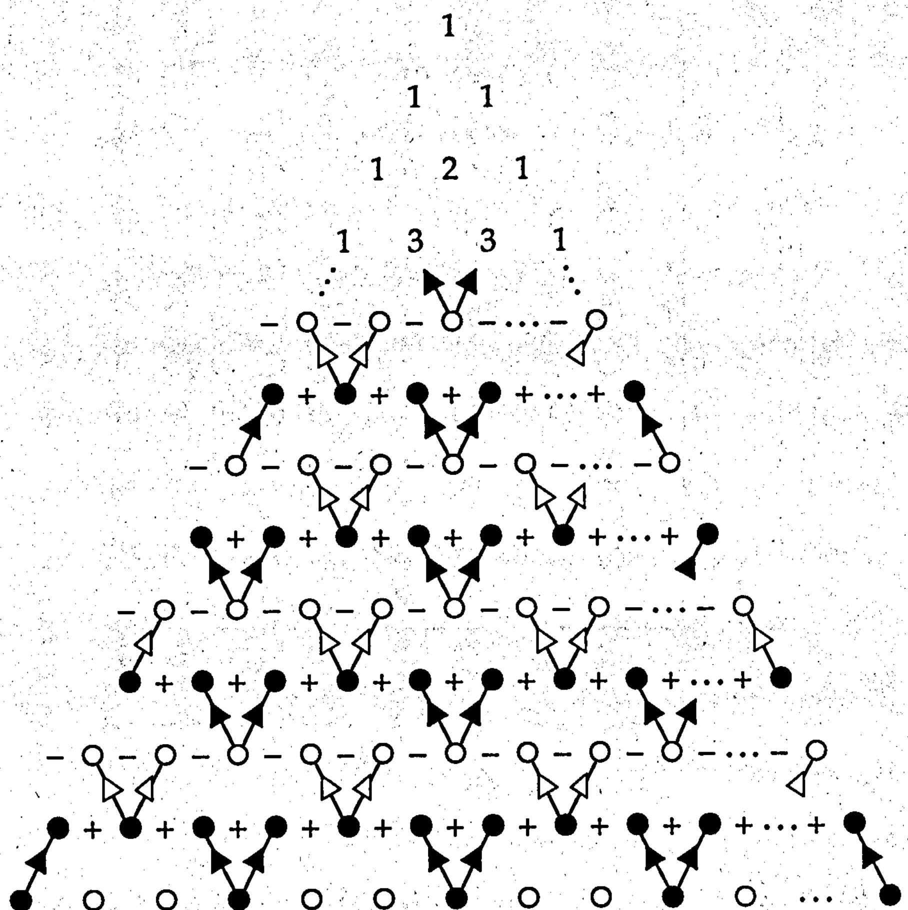
$$\binom{n}{2} = \frac{1}{2}(n^2 - n) = \sum_{i=1}^{n-1} i$$

$$\binom{n+1}{2} = \binom{n}{2} + n$$



—James O. Chilaka

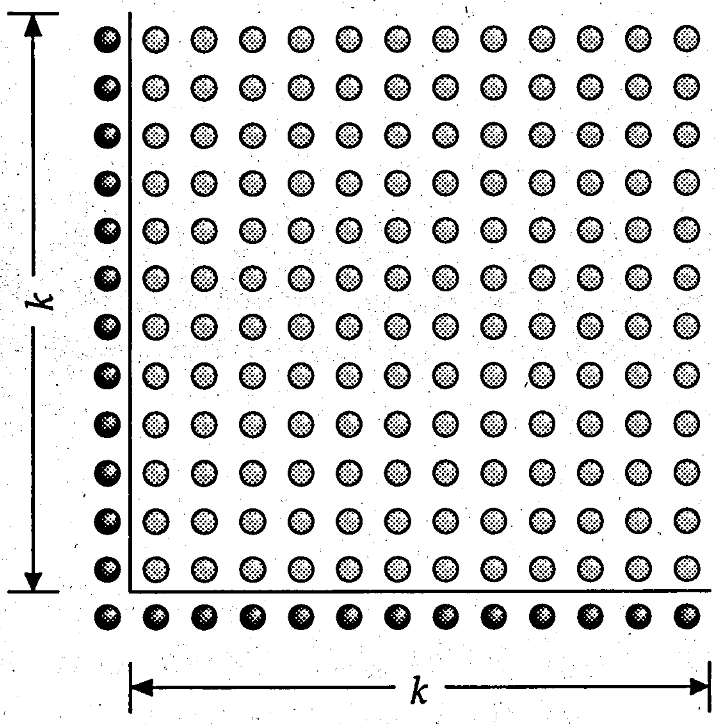
$3\sum_{j=0}^{n}\binom{3n}{3j} = 8^n + 2(-1)^n$, by Inclusion-Exclusion in Pascal's Triangle



$$\sum_{j=0}^{n}\binom{3n}{3j} = \sum_{j=1}^{3n-1}(-1)^{j-1}2^{3n-j} = -2^{3n}\sum_{j=1}^{3n-1}\left(-\frac{1}{2}\right)^{j} = \frac{8^n+2(-1)^n}{3}.$$

—Dean S. Clark

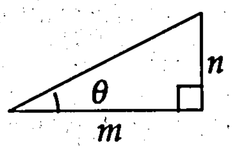
The Existence of Infinitely Many Primitive Pythagorean Triples



$$n^2 = 2k+1 \Rightarrow k^2+n^2 = (k+1)^2 \;\; \& \;\; (k, k+1) = 1$$

—Charles Vanden Eynden

Pythagorean Triples via Double Angle Formulas

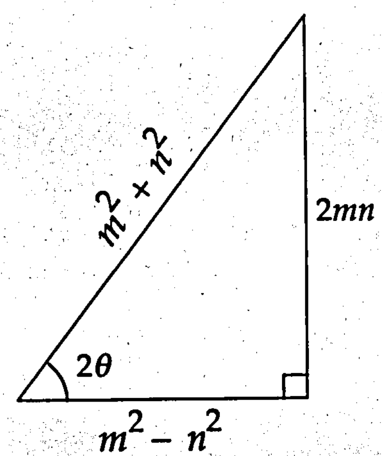


$m > n > 0$

$m, n \in \mathbf{I}$

$$\left\{\begin{aligned} \sin\theta &= \frac{n}{\sqrt{m^2+n^2}} \\ \cos\theta &= \frac{m}{\sqrt{m^2+n^2}} \end{aligned}\right.$$

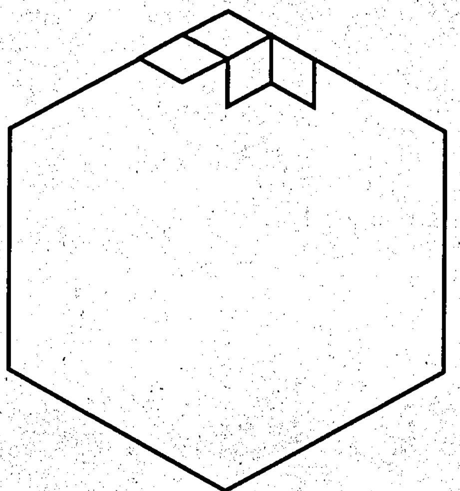
$$\left.\begin{aligned} \sin 2\theta &= \frac{2mn}{m^2+n^2} \\ \cos 2\theta &= \frac{m^2-n^2}{m^2+n^2} \end{aligned}\right\}$$



—David Houston

The Problem of the Calissons

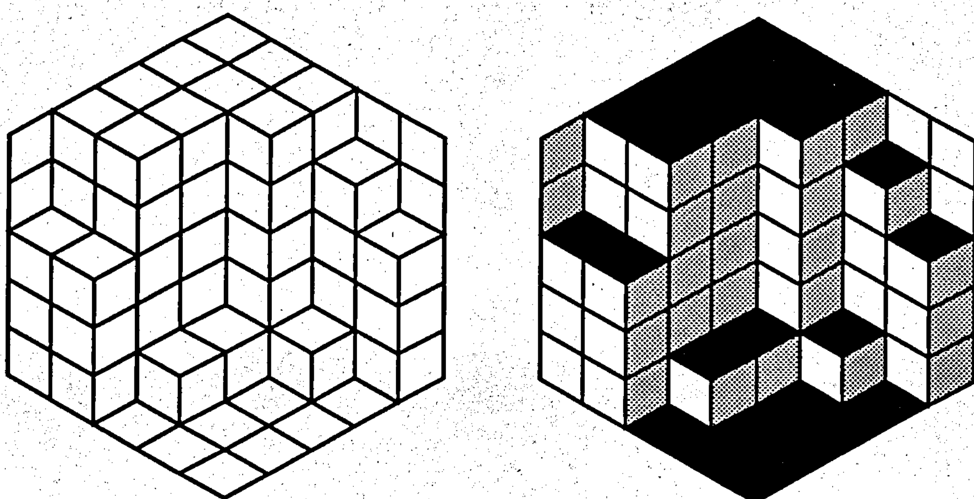
A *calisson* is a French sweet that looks like two equilateral triangles meeting along an edge. Calissons could come in a box shaped like a regular hexagon, and their packing would suggest an interesting combinatorial problem. Suppose a box with side of length n is filled with sweets of sides of length 1. The short diagonal of each calisson in the box is parallel to a pair of sides of the box.



We refer to these three possibilities by saying that a calisson admits three distinct orientations.

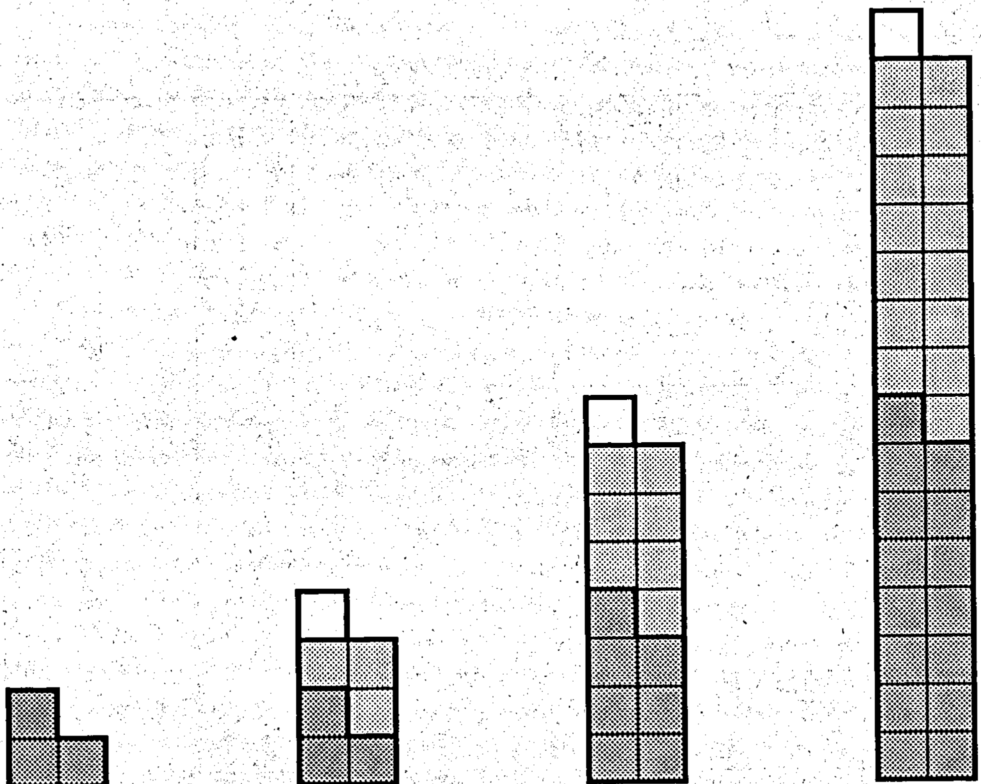
THEOREM: *In any packing, the number of calissons with a given orientation is one-third of the total number of calissons in the box.*

PROOF:



—Guy David and Carlos Tomei

Recursion

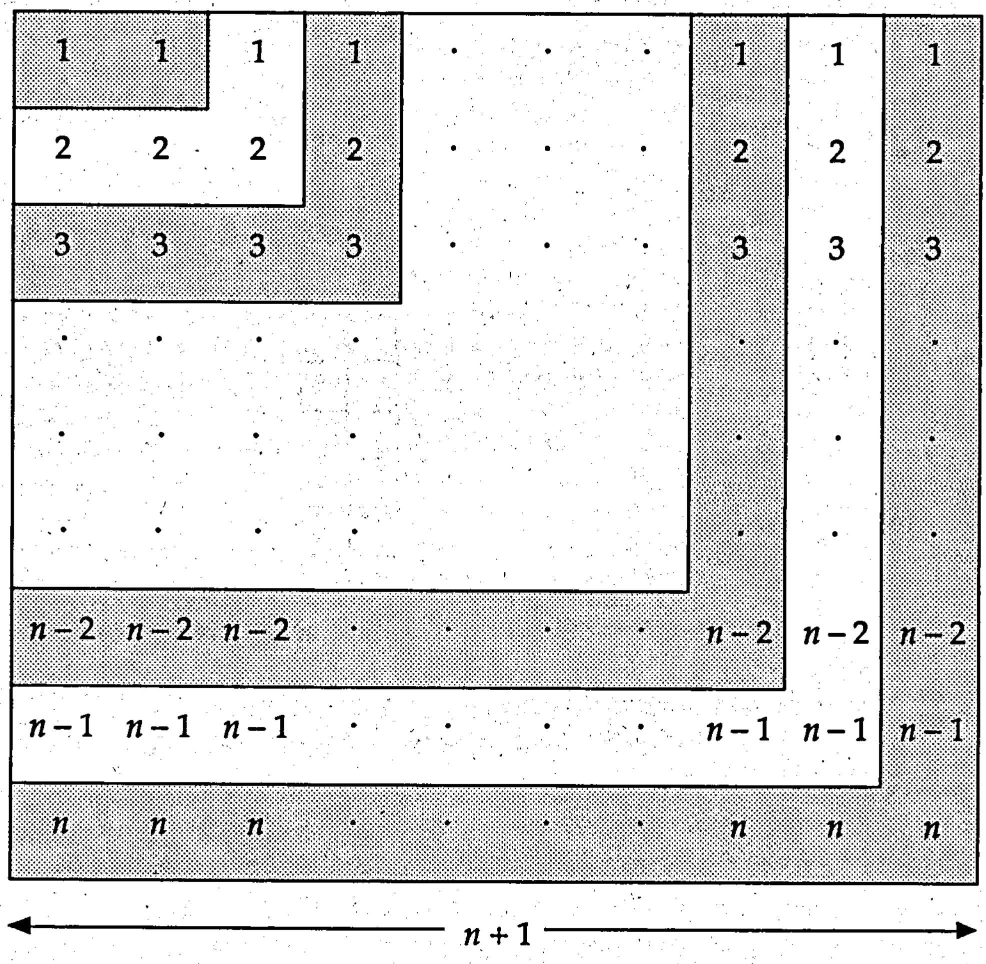


$A_2 = 3$ $A_3 = 2A_2 + 1$ $A_4 = 2A_3 + 1$ $A_5 = 2A_4 + 1$

$$A_2 = 3 \;\&\; A_n = 2A_{n-1} + 1 \;\Leftrightarrow\; A_n = 2(2^{n-1}) - 1 = 2^n - 1$$

—Shirley Wakin

$$\prod_{k=1}^{n} k^k \cdot k! = (n!)^{n+1}$$



—Edward T. H. Wang

Sources

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Geometry & Algebra

3	Howard Eves, *Great Moments in Mathematics (Before 1650)*, The Mathematical Association of America, Washington, 1980, pp. 27-28.
4	Howard Eves, *Great Moments in Mathematics (Before 1650)*, The Mathematical Association of America, Washington, 1980, pp. 29-32.
5	Howard Eves, *Great Moments in Mathematics (Before 1650)*, The Mathematical Association of America, Washington, 1980, pp. 31, 33.
6	Howard Eves, *Great Moments in Mathematics (Before 1650)*, The Mathematical Association of America, Washington, 1980, pp. 29-30.
7	Howard Eves, *Great Moments in Mathematics (Before 1650)*, The Mathematical Association of America, Washington, 1980, pp. 34-36.
8	*College Mathematics Journal*, vol. 17, no. 5 (Nov. 1986), p. 422.
9	*College Mathematics Journal*, vol. 20, no. 1 (Jan. 1989), p. 58.
10	*Mathematics Magazine*, vol. 50, no. 3 (May 1977), p. 162.
11	*Mathematics Magazine*, vol. 48, no. 4 (Sept.-Oct. 1975), p. 198.
12	*College Mathematics Journal*, vol. 21, no. 5 (Nov. 1990), p. 393.
13	*Mathematics Magazine*, vol. 67, no. 5 (Dec. 1994), p. 354.
14	*College Mathematics Journal*, vol. 17, no. 4 (Sept. 1986), p. 338.
15	*Mathematics Magazine*, vol. 62, no. 3 (June 1989), p. 190.
16	*College Mathematics Journal*, vol. 22, no. 5 (Nov. 1991), p. 420.
17	*Mathematics Magazine*, vol. 66, no. 1 (Feb. 1993), p. 13.
18	*Mathematics Magazine*, vol. 65, no. 5 (Dec. 1992), p. 356.
19	*Mathematics Magazine*, vol. 56, no. 2 (March 1983), p. 110.
20	*Mathematics Magazine*, vol. 57, no. 4 (Sept. 1984), p. 231.
21	*Mathematics Magazine*, vol. 64, no. 2 (April 1991), p. 138.
22	*Mathematics Magazine*, vol. 66, no. 3 (June 1993), p. 180.
23	*Mathematics Magazine*, vol. 64, no. 2 (April 1991), p. 114.
24	*Mathematics Magazine*, vol. 68, no. 2 (April 1995), p. 109.
25	*Mathematics Magazine*, vol. 67, no. 4 (Oct. 1994), p. 302.

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Trigonometry, Calculus & Analytic Geometry

29 *Mathematics Magazine*, vol. 64, no. 2 (April 1991), p. 97.
30 *Mathematics Magazine*, vol. 62, no. 5 (Dec. 1989), p. 317.
31 *Mathematics Magazine*, vol. 61, no. 4 (Oct. 1988), p. 259.
32 *Mathematics Magazine*, vol. 63, no. 5 (Dec. 1990), p. 342.
33 *Mathematics Magazine*, vol. 64, no. 2 (April 1992), p. 103.
34 *College Mathematics Journal*, vol. 20, no. 1 (Jan. 1989), p. 51.
35 *American Mathematical Monthly*, vol. 49, no. 5 (May 1942), p. 325.
36 *Mathematics Magazine*, vol. 61, no. 5 (Dec. 1988), p. 281.
37 *Mathematics Magazine*, vol. 61, no. 2 (April 1988), p. 113.
38 *Mathematics Magazine*, vol. 62, no. 4 (Oct. 1989), p. 267.
39 *College Mathematics Journal*, vol. 18, no. 2 (March 1987), p. 141.
40 *Mathematics Magazine*, vol. 42, no. 1 (Jan.-Feb. 1969), pp. 40-41.
41 *College Mathematics Journal*, vol. 16, no. 1 (Jan. 1985), p. 56.
42 Richard Courant, *Differential and Integral Calculus*, p. 219, copyright © 1937, Interscience. Reprinted by permission of John Wiley & Sons, Inc.
43 *College Mathematics Journal*, vol. 18, no. 1 (Jan. 1987), p. 52.
44 *Mathematics Magazine*, vol. 64, no. 3 (June 1991), p. 175.
45 *Mathematics Magazine*, vol. 66, no. 1 (Feb. 1993), p. 39.

Inequalities

49 *Mathematics Magazine*, vol. 50, no. 2 (March 1977), p. 98.
50 *Mathematics Magazine*, vol. 59, no. 1 (Feb. 1986), p. 11.
51 *College Mathematics Journal*, vol. 16, no. 3 (June 1985), p. 208.
52 *College Mathematics Journal*, vol. 19, no. 4 (Sept. 1988), p. 347.
53 *Mathematics Magazine*, vol. 60, no. 3 (June 1987), p. 158.
54 *College Mathematics Journal*, vol. 21, no. 3 (May 1990), p. 227.
55 *College Mathematics Journal*, vol. 20, no. 3 (May 1989), p. 231.
56 *Mathematics Magazine*, vol. 69, no. 1 (Feb. 1996), p. 64.
58 *Mathematics Magazine*, vol. 60, no. 3 (June 1987), p. 165.
59 *Mathematics Magazine*, vol. 64, no. 1 (Feb. 1991), p. 31.
60 *Mathematics Magazine*, vol. 63, no. 3 (June 1990), p. 172.
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Inequalities (continued)

61	II. *Mathematics Magazine*, vol. 67, no. 1 (Feb. 1994), p. 34.
62	*Mathematics Magazine*, vol. 67, no. 5 (Dec. 1994), p. 374.
63	*Mathematics Magazine*, vol. 66, no. 1 (Feb. 1993), p. 65.
64	*Mathematics Magazine*, vol. 67, no. 1 (Feb. 1994), p. 20.
65	*Mathematics Magazine*, in press.
66	*College Mathematics Journal*, vol. 24, no. 2 (March 1993). p. 165.

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69	*Scientific American*, vol. 229, no. 4 (Oct. 1973), p. 114.
70	*Mathematics Magazine*, vol. 57, no. 2 (March 1984), p. 104.
71	Reprinted by permission from *Historical Topics for the Mathematics Classroom*, p. 54, author Bernard H. Gundlach, copyright © 1969 by the National Council of Teachers of Mathematics, Inc.
73	*Mathematics Magazine*, vol. 64, no. 2 (April 1991), p. 103.
74	*Scientific American*, vol 229, no. 4 (Oct. 1973), p. 115.
75	*Mathematics Magazine*, vol. 66, no. 3 (June 1993), p. 166.
76	*Mathematics Magazine*, vol. 59, no. 2 (April 1986), p. 92.
77	*Mathematics Magazine*, vol. 57, no. 2 (March 1984), p. 92.
78	*Scientific American*, vol. 229, no. 4 (Oct. 1973), p. 115. *College Mathematics Journal*, vol. 22, no. 2 (March 1991), p. 124.
79	*College Mathematics Journal*, vol. 20, no. 3 (May 1989), p. 205.
80	*Mathematics Magazine*, vol. 56, no. 2 (March 1983), p. 90.
81	*College Mathematics Journal*, vol. 20, no. 2 (March 1989), p. 123.
82	I. *Mathematics Magazine*, vol. 60, no. 5 (Dec. 1987), p. 291. II. *Mathematics Magazine*, vol. 65, no. 2 (April 1992), p. 90.
83	M. Bicknell & V. E. Hoggatt, Jr. (eds.), *A Primer for the Fibonacci Numbers*, The Fibonacci Association, San Jose, 1972, p. 147.
84	*Mathematical Gazette*, vol. 49, no. 368 (May 1965), p. 199.
85	*Mathematics Magazine*, vol. 50, no. 2 (March 1977), p. 74.
86	*Mathematics Magazine*, vol. 58, no. 1 (Jan. 1985), p. 11.
87	*Mathematics Magazine*, vol. 62, no. 4 (Oct. 1989), p. 259. *Mathematical Gazette*, vol. 49, no. 368 (May 1965), p. 200.
88	*Mathematics Magazine*, vol. 63, no. 3 (June 1990), p. 178.
89	*Mathematics Magazine*, vol. 62, no. 5 (Dec. 1989), p. 323.
90	*Mathematics Magazine*, vol. 65, no. 3 (June 1992), p. 185.

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91	*Mathematics Magazine*, vol. 68, no. 5 (Dec. 1995), p. 371.
92	*Mathematics Magazine*, vol. 65, no. 1 (Feb. 1992), p. 55.
93	*Mathematics Magazine*, vol. 66, no. 5 (Dec. 1993), p. 329.
94	*Mathematics Magazine*, vol. 63, no. 5 (Dec. 1990), p. 314.
95	*College Mathematics Journal*, vol. 23, no. 5 (Nov. 1992), p. 417.
96	Richard K. Guy, written communication.
97	*Mathematics Magazine*, vol. 62, no. 1 (Feb. 1989), p. 27.
98	*Mathematics Magazine*, vol. 62, no. 2 (April 1989), p. 96.
99	*College Mathematics Journal*, vol. 18, no. 4 (Sept. 1987), p. 318.
100	*Mathematics Magazine*, vol. 66, no. 5 (Dec. 1993), p. 325.
101	*Mathematics Magazine*, vol. 58, no. 4 (Sept. 1985), pp. 203, 236.
102	*Mathematics Magazine*, vol. 63, no. 1 (Feb. 1990), p. 25.
103	*Mathematics Magazine*, vol. 55, no. 2 (March 1982), p. 97.
104	Richard K. Guy, written communication.
105	*Mathematics Magazine*, vol. 67, no. 4 (Oct. 1994), p. 293.
106	Richard K. Guy, written communication.
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108	*Mathematics Magazine*, vol. 63, no. 4 (Oct. 1990), p. 225.
109	*American Mathematical Monthly*, vol. 95, no. 8 (Oct. 1988), pp. 701, 709.
110	*Mathematics Magazine*, vol. 66, no. 5 (Dec. 1993), p. 316.
111	*Mathematics Magazine*, vol. 63, no. 5 (Dec. 1990), p. 349.
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115	*Mathematics Magazine*, vol. 68, no. 1 (Feb. 1995), p. 41.
116	*Mathematics Magazine*, vol. 67, no. 5 (Dec. 1994), p. 379.
117	*Mathematics Magazine*, vol. 66, no. 3 (June 1993), p. 179.
118	*Mathematics Magazine*, vol. 54, no. 4 (Sept. 1981), p. 201.
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121	*Mathematics Magazine*, vol. 67, no. 3 (June 1994), p. 230.
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123	*Mathematics Magazine*, vol. 67, no. 3 (June 1994), p. 209.
124	*Mathematics Magazine*, vol. 62, no. 5 (Dec. 1989), pp. 332-333.
126	*Mathematics Magazine*, vol. 65, no. 5 (Dec. 1992), p. 338.
127	*Mathematics Magazine*, vol. 64, no. 3 (June 1991), p. 167.

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128 *American Mathematical Monthly*, vol. 94, no. 6 (June-July 1988), pp. 541-42.
129 *Mathematics Magazine*, vol. 65, no. 2 (April 1992), p. 136.
130 *Mathematics Magazine*, vol. 64, no. 4 (Oct. 1991), p. 241.

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133 *Mathematics Magazine*, vol. 58, no. 2 (March 1985), p. 107.
134 *Mathematics Magazine*, vol. 64, no. 5 (Dec. 1991), p. 339.
135 *Mathematics Magazine*, vol. 61, no. 5 (Dec. 1988), p. 294.
136 *Mathematics Magazine*, vol. 60, no. 2 (April 1987), p. 89.
137 *Mathematics Magazine*, vol. 61, no. 2 (April 1988), p. 98.
138 *Mathematics Magazine*, vol. 52, no. 4 (Sept. 1979), p. 206.
139 *Mathematics Magazine*, vol. 63, no. 1 (Feb. 1990), p. 29.
140 *Elementary Number Theory*, Random House, New York, 1987, p. 227.
141 *Mathematics Magazine*, vol. 67, no. 3 (June 1994), p. 187.
142 *American Mathematical Monthly*, vol. 96, no. 5 (May 1989), pp. 429-30.
143 *Mathematics Magazine*, vol. 62, no. 3 (June 1989), p. 172.
144 *College Mathematics Journal*, vol. 20, no. 2 (March 1989), p. 152.

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Technical Note

The manuscript for this book was edited and printed using Microsoft® Word 5.1 on an Apple® Macintosh™ IIfx computer. The graphics were produced in Claris™ MacDraw® Pro 1.5v1. The text is set in the Palatino font, with special characters in the Symbol font. Many of the displayed equations were produced with MacΣqn™ 3.0 from Software for Recognition Technologies. The manuscript was printed on an Apple® Laserwriter® II NTX.

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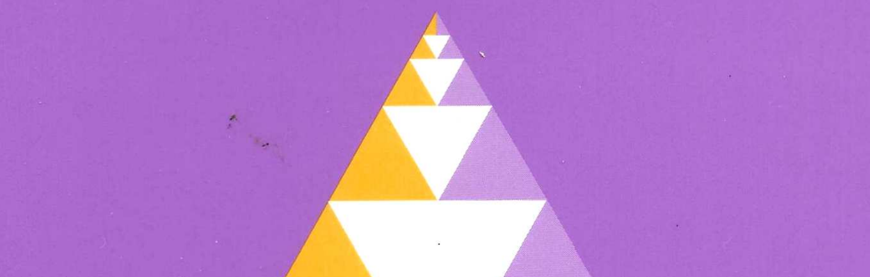

$$\frac{1}{4}+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^3+\cdots=\frac{1}{3}$$

ROGER B. NELSEN

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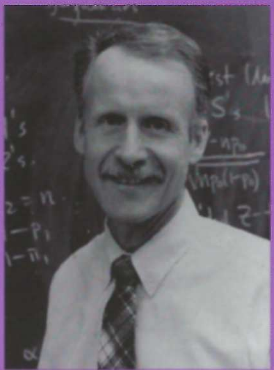

Roger Nelsen received his Ph.D. in mathematics from Duke University. Since 1969 he has taught at Lewis & Clark College, where he is professor of mathematics. He currently chairs the editorial board for the Mathematical Association of America's series of Problem Books, and served for ten years as an associate editor of the *College Mathematics Journal*. He has published expository and research articles in the *American Mathematical Monthly, Mathematics Magazine,* the *College Mathematics Journal, Statistics & Probability Letters,* the *Journal of Multivariate Analysis,* and the *Journal of Nonparametric Statistics*. His other books are *Proofs Without Words: Exercises in Visual Thinking* (MAA, 1993) and *An Introduction to Copulas* (Springer, 1999).

What are "proofs without words?" Many would argue that they are not really "proofs" (nor, for that matter, are many "without words," on account of equations which often accompany them). Like its predecessor *Proofs Without Words*, published by the MAA in 1993, this book is a collection of pictures or diagrams that help the reader see *why* a particular mathematical statement may be true, and also to see *how* one might begin to go about proving it true. The emphasis is on providing visual clues to the observer to stimulate mathematical thought.

Proofs without words have been around for a long time. In this volume you find modern renditions of proofs without words from ancient China, tenth century Arabia, and Renaissance Italy. While the majority of the proofs without words in this book originally appeared in journals published by the MAA, others first appeared in journals published by other organizations in the US and abroad, and on the World Wide Web.

The proofs in this collection are arranged by topic into five chapters. Although the proofs without words are presented primarily for the enjoyment of the reader, teachers will want to use them with students at many levels—in precalculus courses in high school, in college courses in calculus, number theory and combinatorics, and in pre-service and in-service classes for teachers.

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Proofs Without Words II

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Library of Congress Catalog Card Number 00-108051

ISBN 978-0-88385-721-2

Printed in the United States of America

Current printing (last digit):
10 9 8 7 6 5 4

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This book is dedicated to the memory of my parents,

Ann Bain Nelsen and Howard Ernest Nelsen.

Contents

Introduction

Proofs really aren't there to convince you that something is true—they're there to show you why it is true.

—Andrew Gleason

A good proof is one that makes us wiser.

—Yu. I. Manin

Much research for new proofs of theorems already correctly established is undertaken simply because the existing proofs have no aesthetic appeal. There are mathematical demonstrations that are merely convincing; to use a phrase of the famous mathematical physicist, Lord Rayleigh, they "command assent." There are other proofs "which woo and charm the intellect. They evoke delight and an overpowering desire to say, Amen, Amen." An elegantly executed proof is a poem in all but the form in which it is written.

—Morris Kline

What are "proofs without words?" As you will see from this second collection, the question does not have a simple, concise answer (the first collection, *Proofs Without Words: Exercises in Visual Thinking*, was published by the Mathematical Association of America in 1993). Generally, proofs without words (PWWs) are pictures or diagrams that help the reader see *why* a particular mathematical statement may be true, and also to see *how* one might begin to go about proving it true. In some, an equation or two may appear in order to guide the observer in this process. The emphasis is, however, clearly on providing visual clues to the observer to stimulate mathematical thought.

Proofs without words are regular features in the journals published by the Mathematical Association of America. PWWs began to appear in *Mathematics Magazine* about 1975, and in *The College Mathematics Journal* about ten year later. But proofs without words are not recent innovations—they have been around for a very long time. In this volume you find modern renditions of PWWs from ancient China, tenth century Arabia, and Renaissance Italy. Proofs without words also now appear in journals published by other organizations in the U.S. and abroad, and on the World Wide Web.

Of course, some argue that PWWs are not really "proofs" (nor, for that matter, are they "without words," on account of equations which often accompany a PWW). In his recent book *Philosophy of Mathematics: An Introduction to the World of Proofs and Pictures* (Routledge, London, 1999), James Robert Brown notes:

> "Mathematicians, like the rest of us, cherish clever ideas; in particular they delight in an ingenious picture. But this appreciation does not overwhelm a prevailing skepticism. After all, a diagram is—at best—just a special case and so can't establish a general theorem. Even worse, it can be downright misleading. Though not universal, the prevailing attitude is that pictures are really no more than heuristic devices; they are psychologically suggestive and pedagogically important—but they *prove* nothing. I want to oppose this view and to make a case for pictures having a legitimate role to play as evidence and justification—a role well beyond the heuristic. In short, pictures can prove theorems."

In my introduction to the first collection of PWWs, I suggested that teachers might want to share the PWWs with their students. Several readers of the first collection responded to my request for information about ways in which PWWs are being used in the classroom. Respondents commented on using PWWs with classes at all levels—precalculus courses in high school, college courses in calculus, number theory, and combinatorics, and pre-service and in-service classes for teachers. PWWs appear to be used frequently to supplement or even replace "textbook" proofs, for example, for the Pythagorean theorem or the formulas for sums of integers, squares, and cubes. Other uses range from regular assignments, extra-credit problems, in-class presentations by students, and even term papers and projects.

I should note that this collection, like the first, is necessarily incomplete. It does not include all PWWs which have appeared in print since the first collection was published in 1993, nor all of those which I overlooked in compiling the first book. As readers of the Association's journals are well aware, new PWWs appear in print rather frequently, and they also appear now on the World Wide Web in formats superior to print, involving motion and viewer interaction.

I hope that the readers of this collection will find enjoyment in discovering or rediscovering some elegant visual demonstrations of certain mathematical ideas; that teachers will share them with their students; and that all will find stimulation and encouragement to create new proofs without words.

Acknowledgment. I would like to express my appreciation and gratitude to all those individuals who have contributed proofs without words to the mathematical literature; see the *Index of Names* on pp. 127-128. Without them this collection simply would not exist. Thanks to Andy Sterrett and the members of the editorial board of Classroom Resource Materials for their careful reading of an earlier draft of the book, and for their many helpful suggestions. I would also like to thank Elaine Pedreira, Beverly Ruedi, and Don Albers of the MAA's publication staff for their encouragement, expertise, and hard work in preparing this book for publication.

Roger B. Nelsen
Lewis & Clark College
Portland, Oregon

Notes

1. The illustrations in this collection were redrawn to create a uniform appearance. In a few instances titles were changed, and shading or symbols were added or deleted for clarity. Any errors resulting from that process are entirely my responsibility.

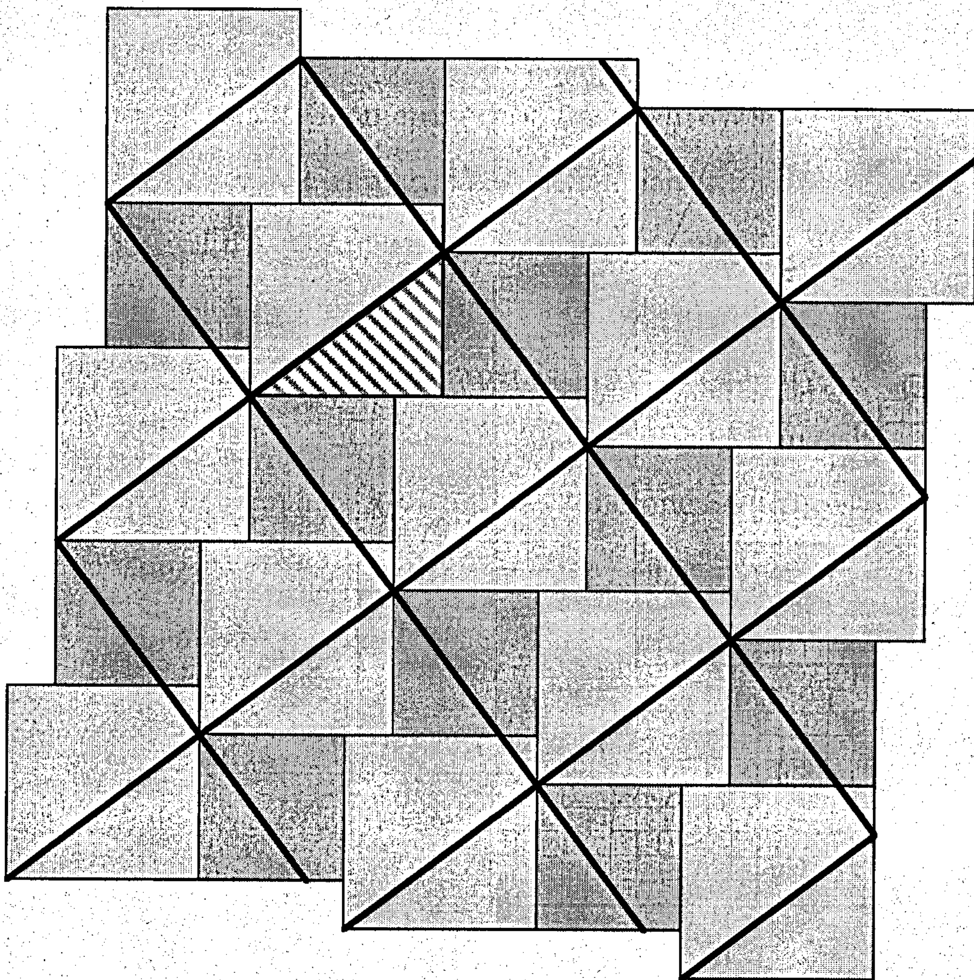
2. Roman numerals are used in the titles of some PWWs to distinguish multiple PWWs of the same theorem—and the numbering is carried over from *Proofs Without Words*. So, for example, since there are six PWWs of the Pythagorean Theorem in *Proofs Without Words*, the first in this collection carries the title "The Pythagorean Theorem VII."

3. Several PWWs in this collection are presented in the form of "solutions" to problems from mathematics contests such as the William Lowell Putnam Mathematical Competition and the Canadian Mathematical Olympiad. It is quite doubtful that such "solutions" would have garnered many points in those contests, as contestants are advised in, for example, the Putnam Competition that "all the necessary steps of a proof must be shown clearly to obtain full credit."

4. The three quotations at the beginning of the Introduction are from *Out of the Mouths of Mathematicians* by Rosemary Schmalz, (Mathematical Association of America, Washington, 1993), pp. 75, 62, and 135-136.

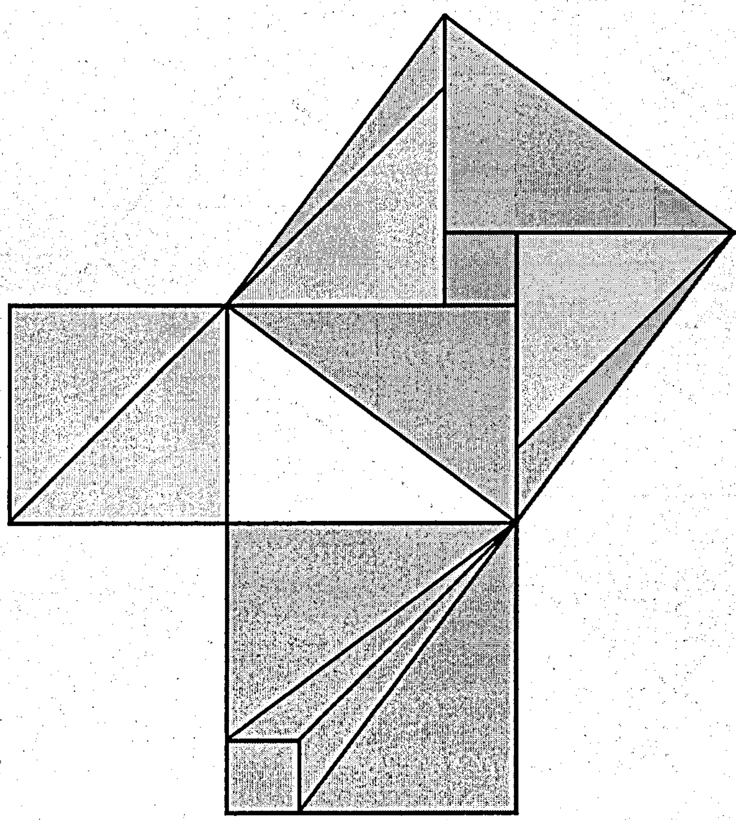
Geometry & Algebra

The Pythagorean Theorem VII



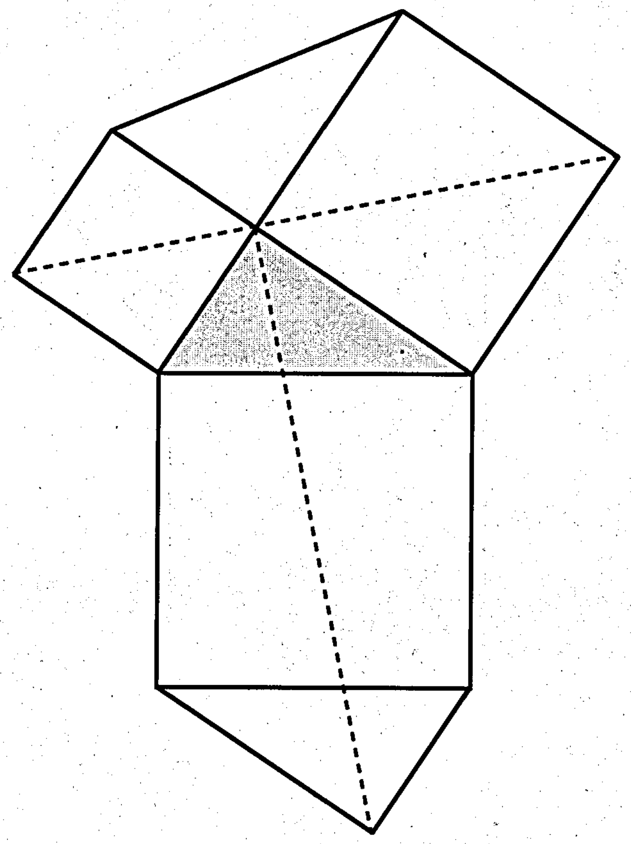
—Annairizi of Arabia (circa A.D. 900)

The Pythagorean Theorem VIII



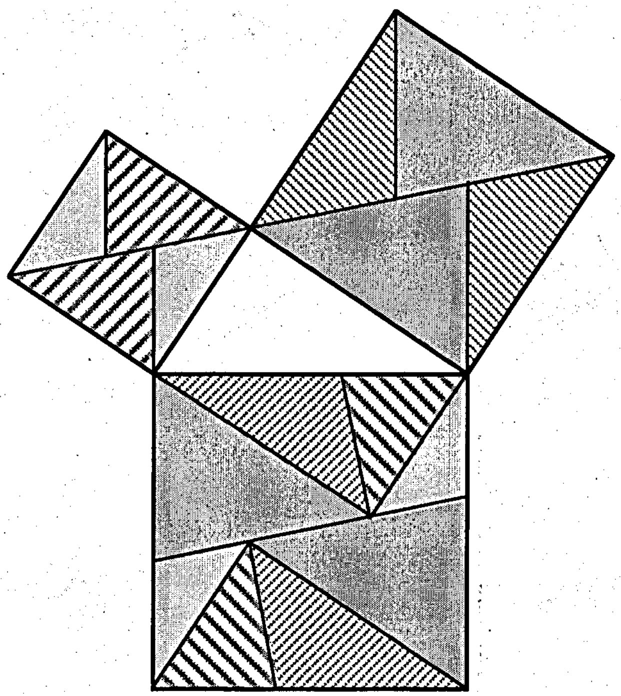
—Liu Hui (3rd century A.D.)

The Pythagorean Theorem IX



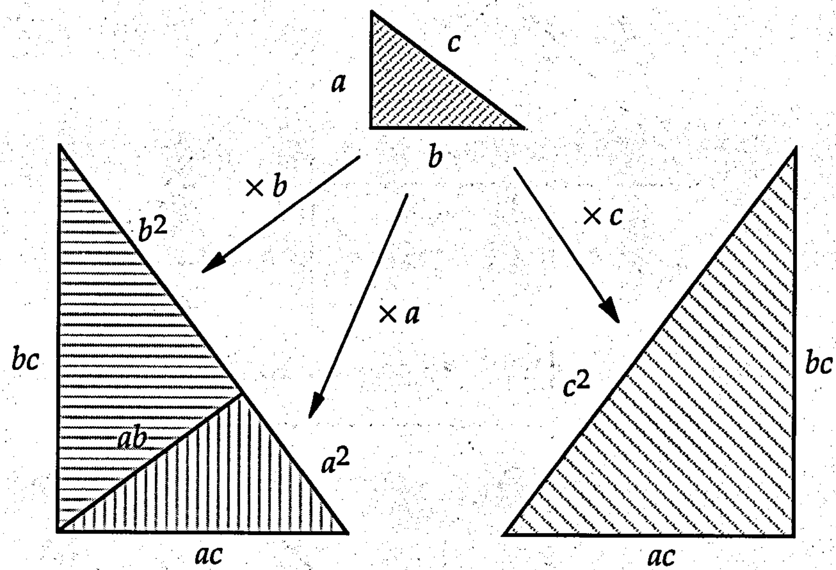
—Leonardo da Vinci (1452-1519)

The Pythagorean Theorem X



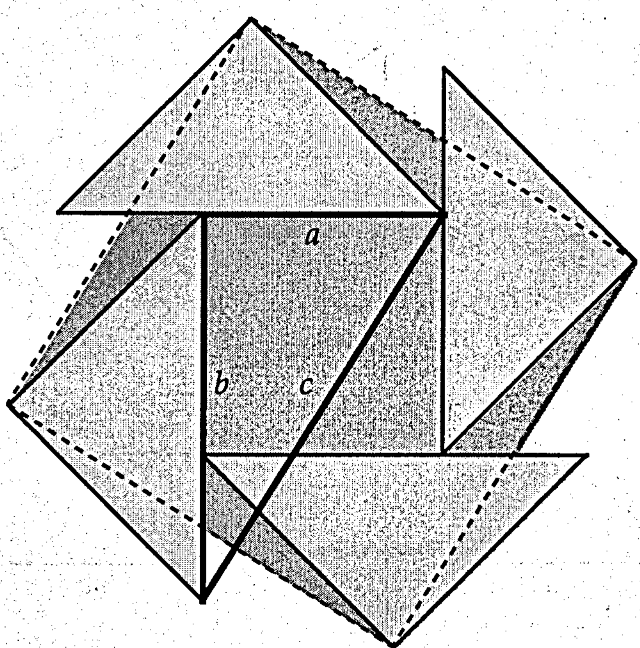
—J. E. Böttcher

The Pythagorean Theorem XI



—Frank Burk

The Pythagorean Theorem XII

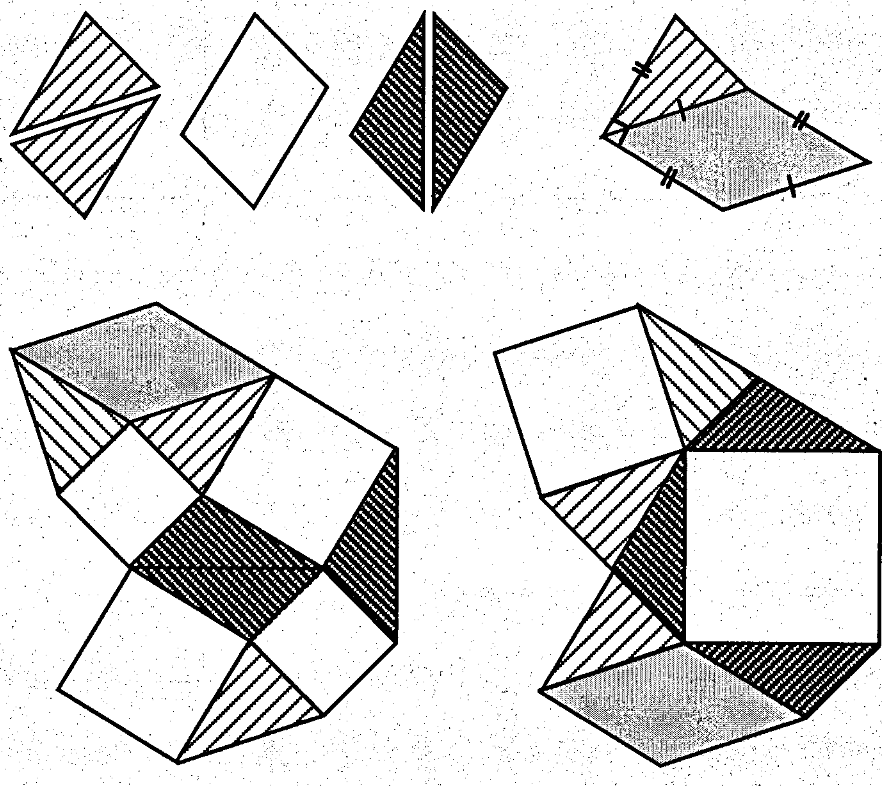


$$a^2 + b^2 = c^2$$

—Poo-sung Park

A Generalization from Pythagoras

The sum of the area of two squares, whose sides are the lengths of the two diagonals of a parallelogram, is equal to the sum of the areas of four squares, whose sides are its four sides.

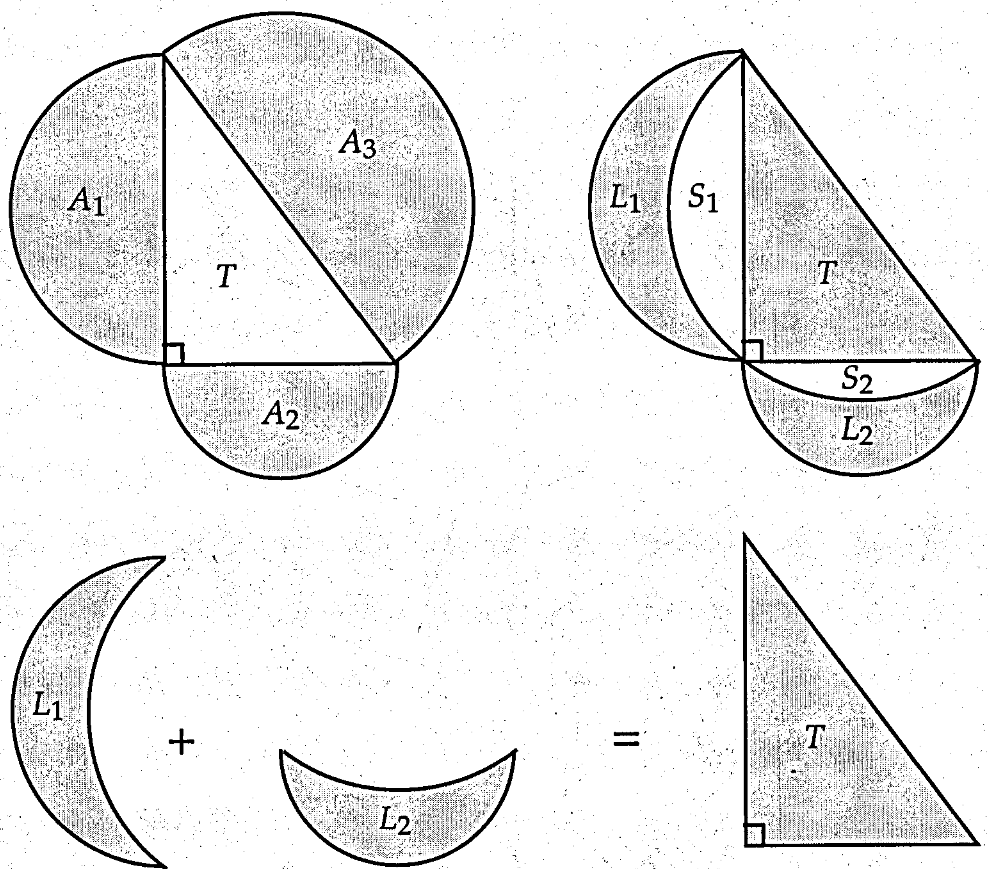


COROLLARY: The Pythagorean Theorem (when the parallelogram is a rectangle).

—David S. Wise

A Theorem of Hippocrates of Chios (circa B.C. 440)

The combined area of the lunes constructed on the legs of a given right triangle is equal to the area of the triangle.



$$A_1 + A_2 = A_3$$

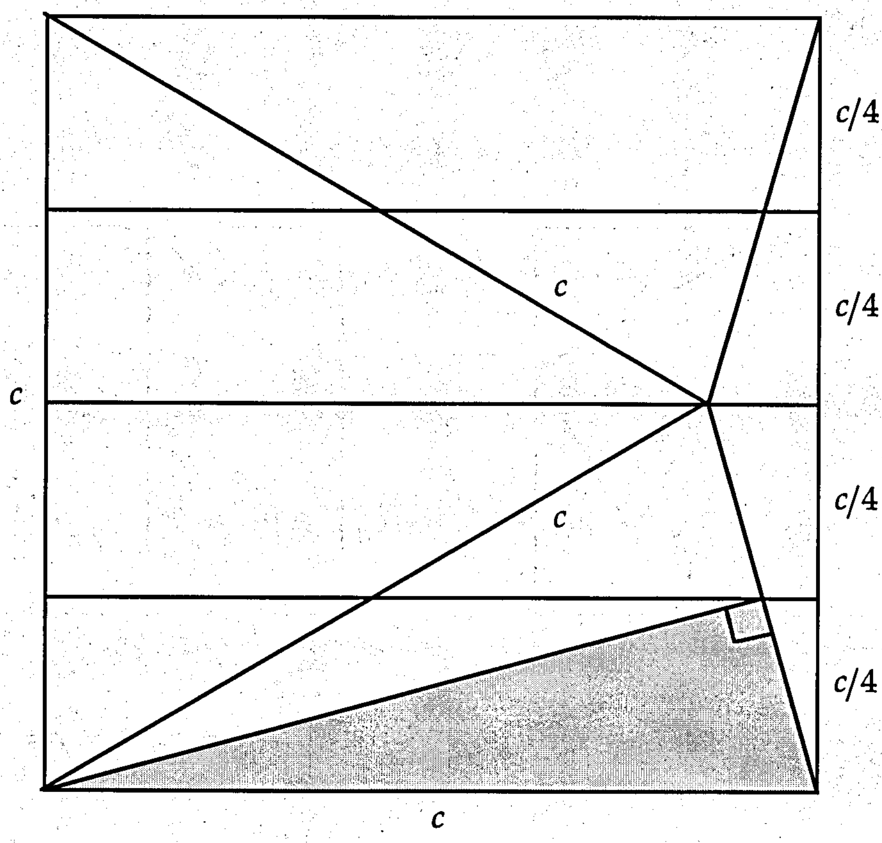
$$(L_1 + S_1) + (L_2 + S_2) = T + S_1 + S_2$$

$$L_1 + L_2 = T$$

—Eugene A. Margerum
and Michael M. McDonnell

The Area of a Right Triangle with Acute Angle $\pi/12$

The area of a right triangle is $\frac{1}{8}(\text{hypotenuse})^2$ if and only if one acute angle is $\pi/12$.



—Klara Pinter

A Right Triangle Inequality

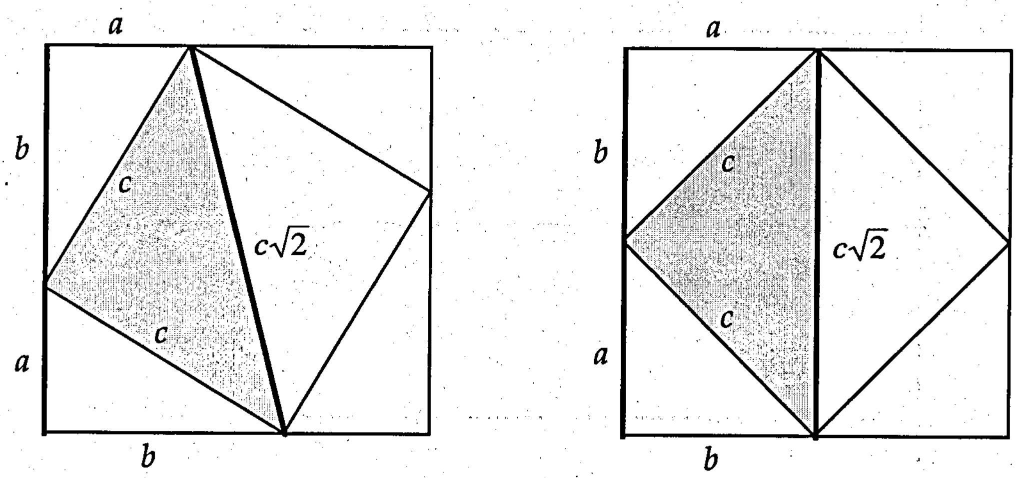
(Problem 3, The Canadian Mathematical Olympiad, 1969)

Let c be the length of the hypotenuse of a right triangle whose other two sides have lengths a and b. Prove that

$$a+b \leq c\sqrt{2}.$$

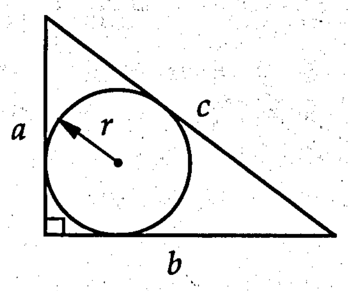
When does equality hold?

SOLUTION:



$a+b \leq c\sqrt{2}$ $\qquad$ $a+b=c\sqrt{2} \Leftrightarrow a=b$

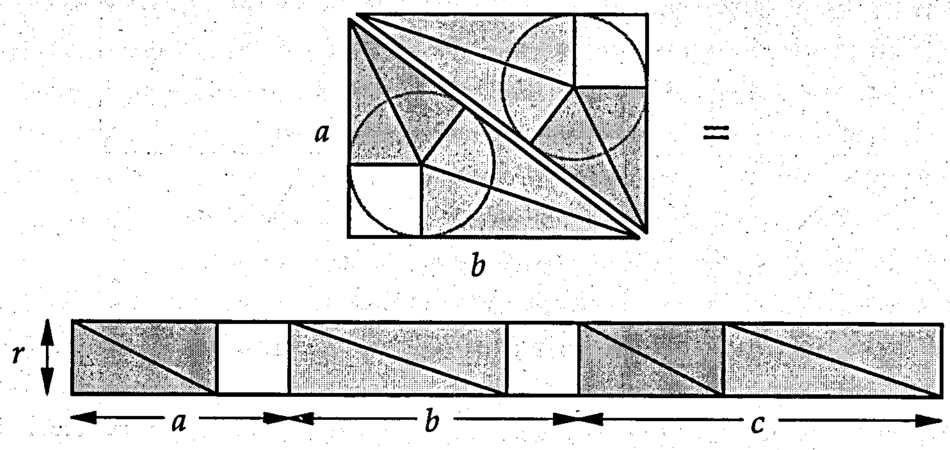
The Inradius of a Right Triangle



I. $r = \dfrac{ab}{a+b+c}$

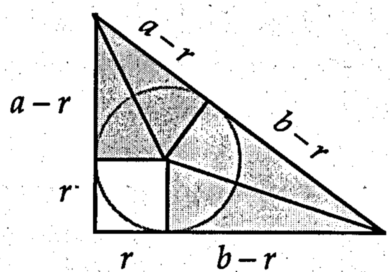
II. $r = \dfrac{a+b-c}{2}$

I. $ab = r(a+b+c)$



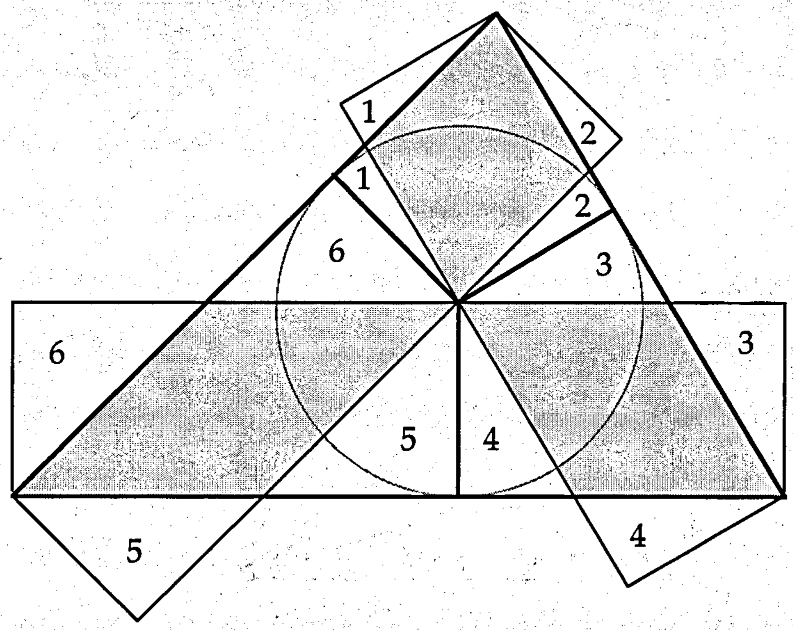
—Liu Hui (3rd century A.D.)

II. $c = a + b - 2r$



The Product of the Perimeter of a Triangle and Its Inradius is Twice the Area of the Triangle

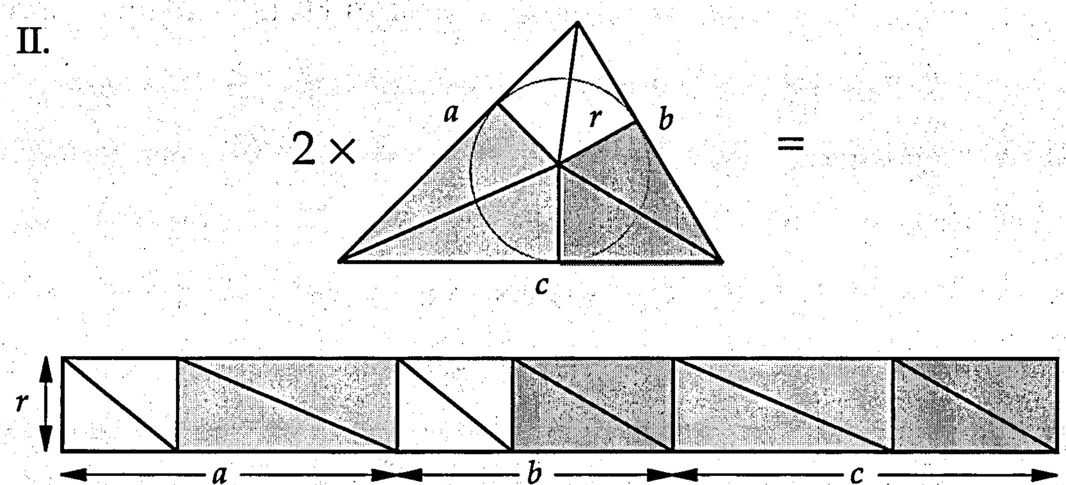
I.



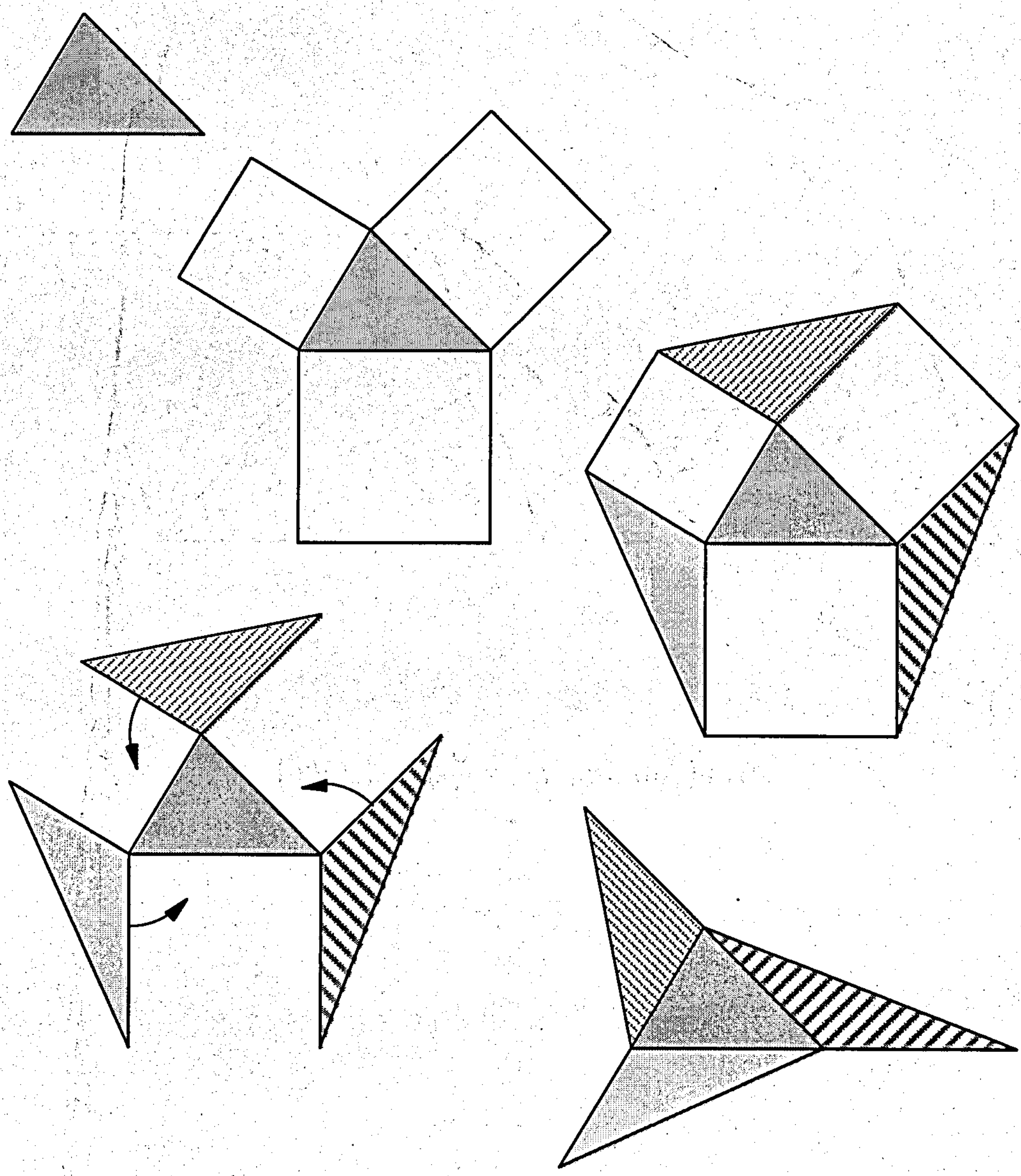
NOTE: Regions bearing the same number are equal in area.

—Grace Lin

II.

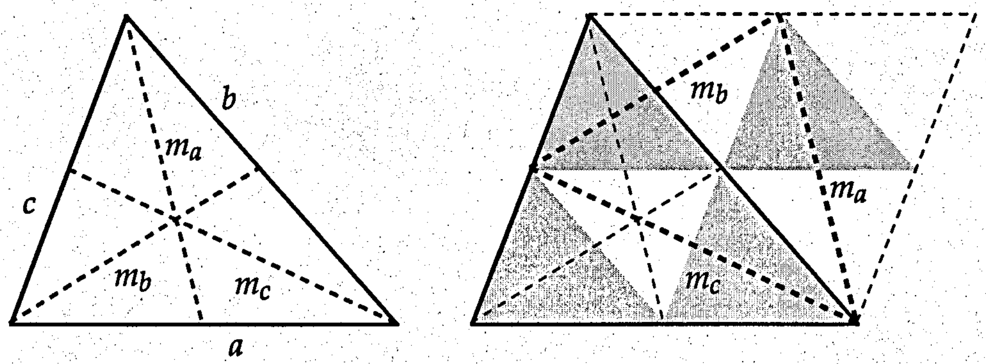


Four Triangles with Equal Area



—Steven L. Snover

The Triangle of Medians Has Three-Fourths the Area of the Original Triangle

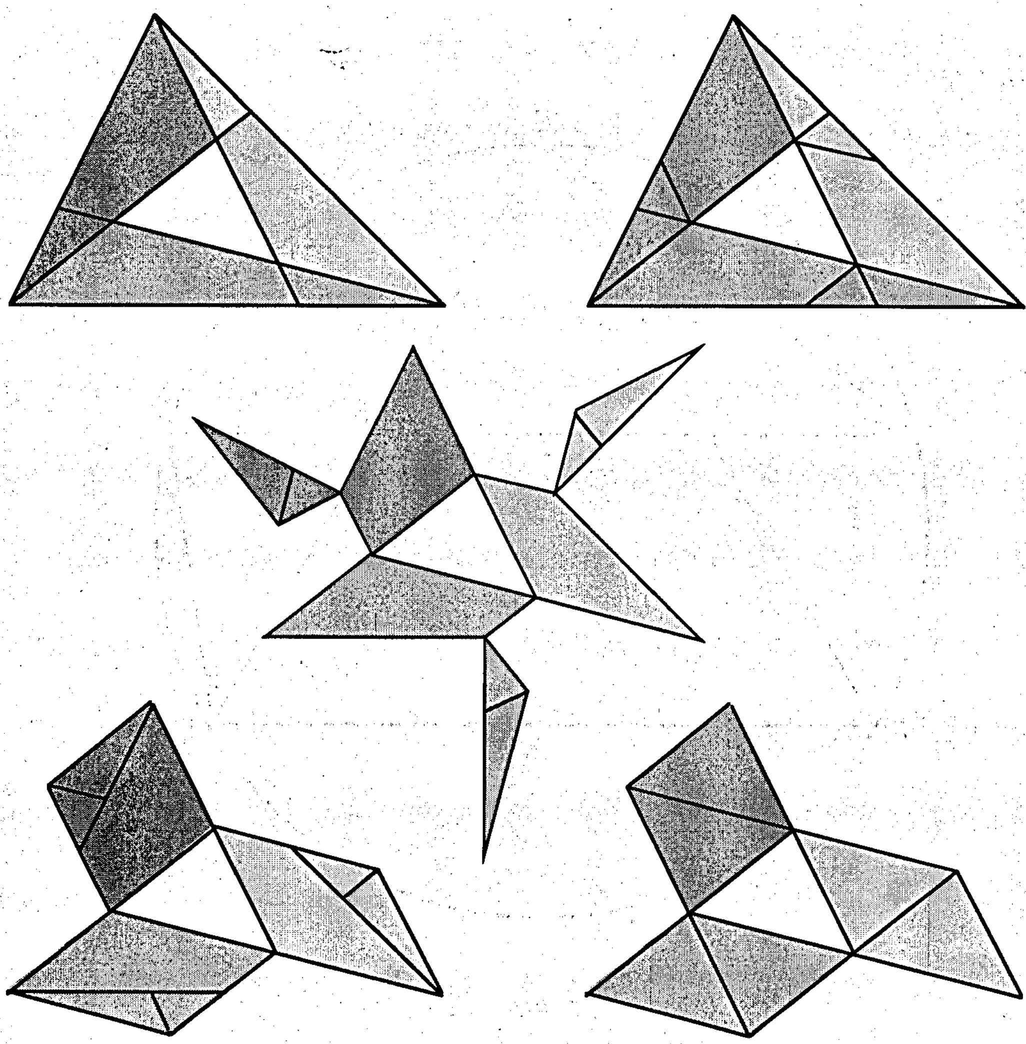


$$\text{area}(\Delta m_a m_b m_c) = \frac{3}{4}\text{area}(\Delta abc)$$

—Norbert Hungerbühler

Heptasection of a Triangle

If the one-third points on each side of a triangle are joined to opposite vertices, the resulting central triangle is equal in area to one-seventh that of the initial triangle.



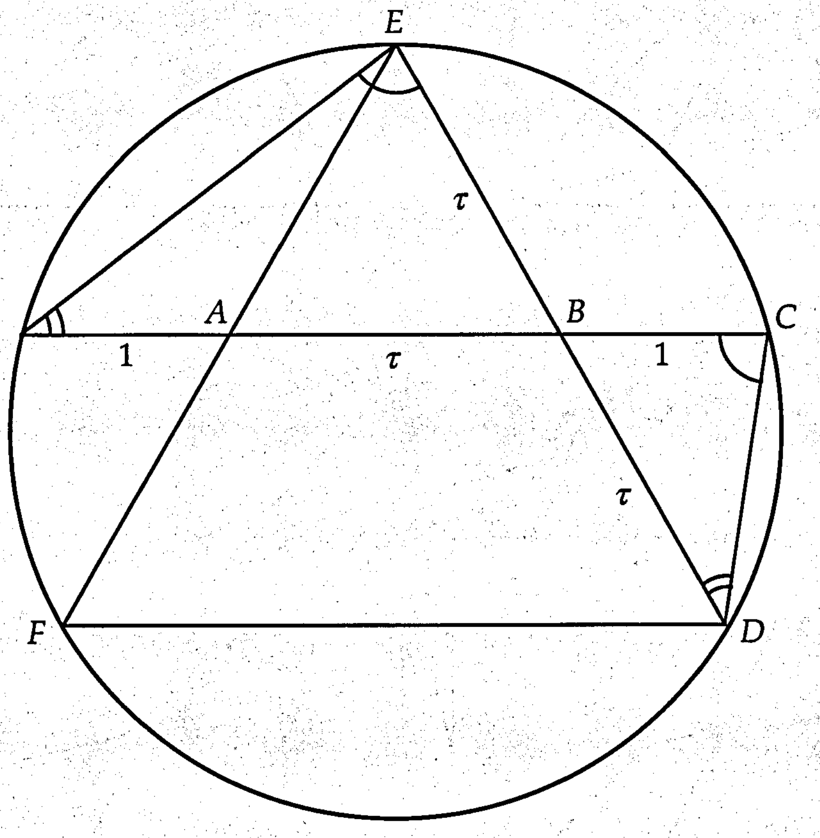
—William Johnston
and Joe Kennedy

A Golden Section Problem from the *Monthly*

(Problem E3007, *American Mathematical Monthly*, 1983, p. 482)

Let A and B be the midpoints of the sides EF and ED of an equilateral triangle DEF. Extend AB to meet the circumcircle (of DEF) at C. Show that B divides AC according to the golden section.

SOLUTION:

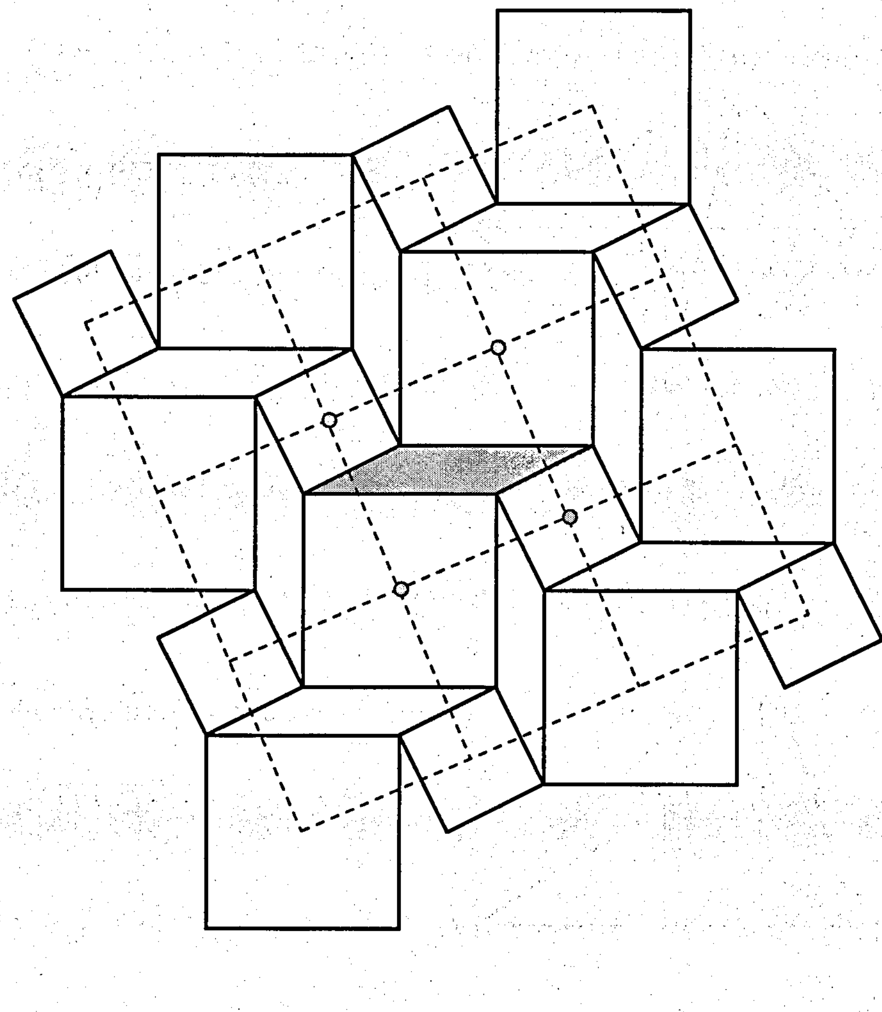


$$\tau^2 = \tau + 1$$

—Jan van de Craats

Tiling with Squares and Parallelograms

If squares are constructed externally on the sides of a parallelogram, their centers form a square.

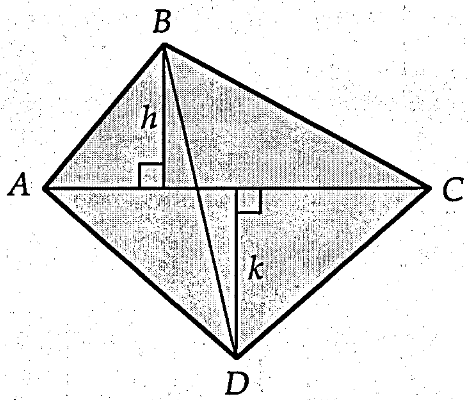


—Alfinio Flores

The Area of a Quadrilateral I

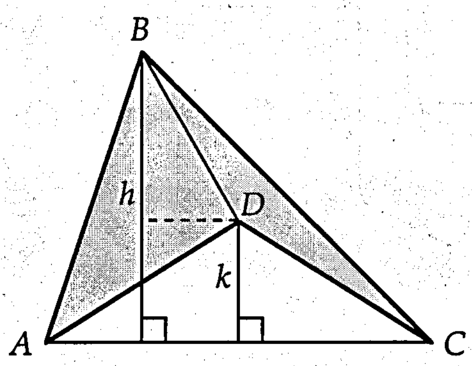
The area of a quadrilateral is less than or equal to half the product of the lengths of its diagonals, with equality if and only if the diagonals are perpendicular.

I. Convex quadrilaterals



$$\text{Area} = \frac{1}{2}\overline{AC}\cdot(h+k)$$
$$\leq \frac{1}{2}\overline{AC}\cdot\overline{BD}$$

II. Concave quadrilaterals



$$\text{Area} = \frac{1}{2}\overline{AC}\cdot(h-k)$$
$$\leq \frac{1}{2}\overline{AC}\cdot\overline{BD}$$

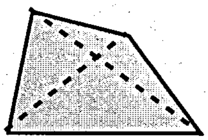
—David B. Sher, Ronald Skurnick,
and Dean C. Nataro

The Area of a Quadrilateral II

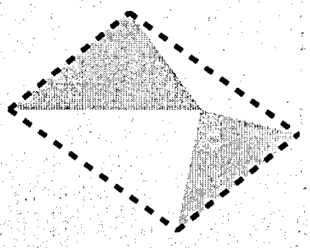
The area of a quadrilateral Q is equal to one-half the area of a parallelogram P whose sides are parallel to and equal in length to the diagonals of Q.

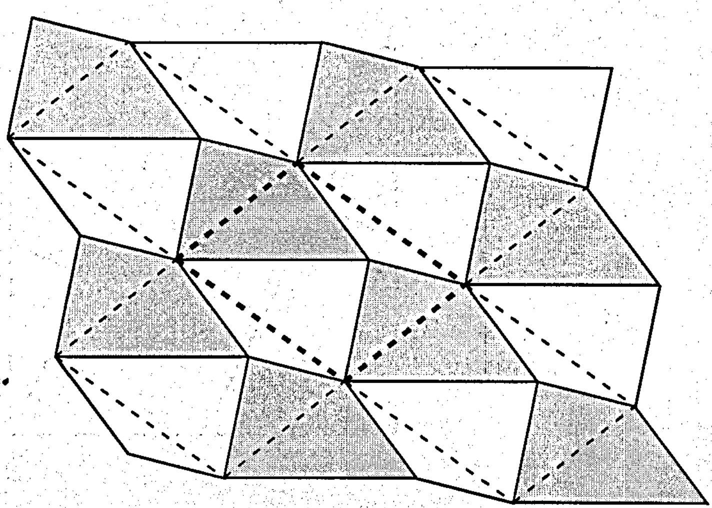
I. Q convex

Q:



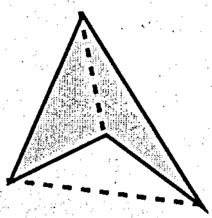
P:



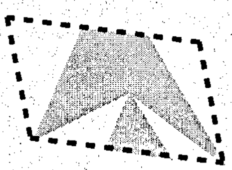


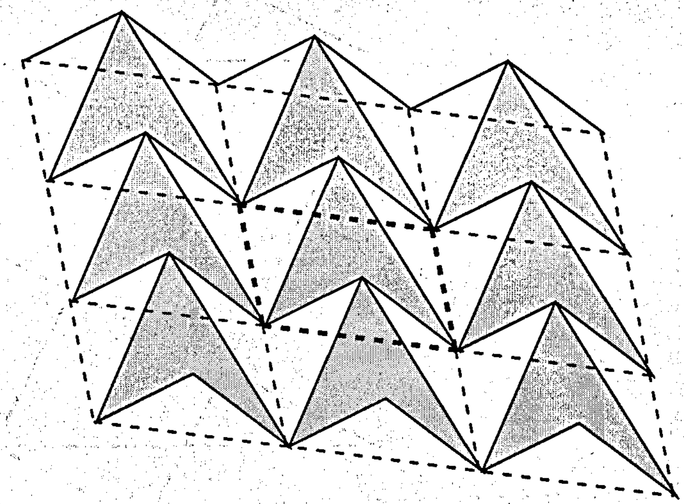
II. Q concave

Q:



P:

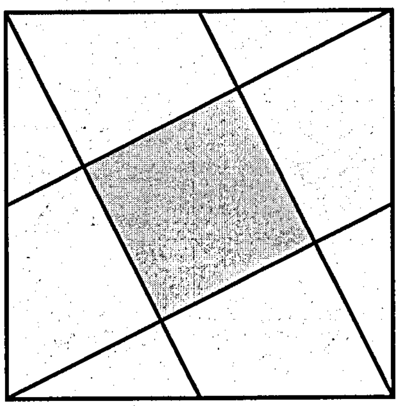


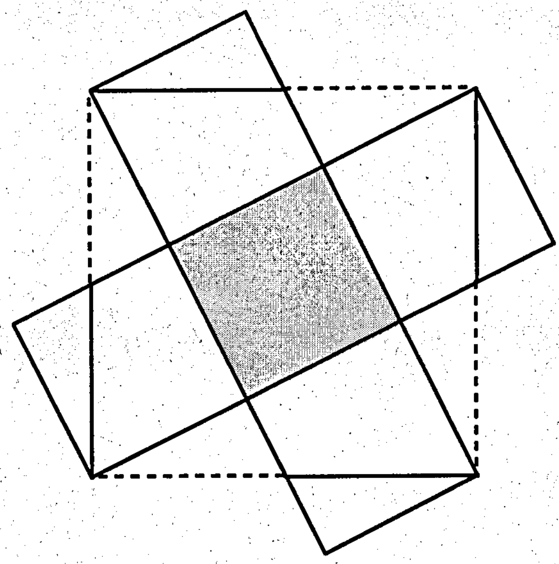


$$\text{area}(Q) = \frac{1}{2}\text{area}(P)$$

A Square Within a Square

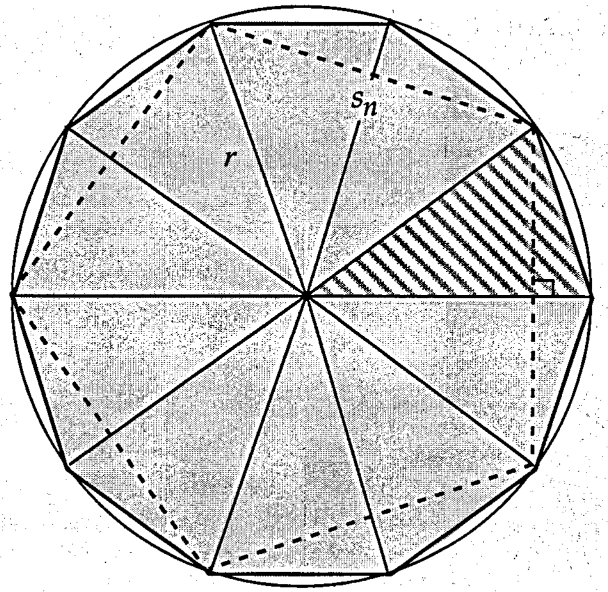
If lines from the vertices of a square are drawn to the midpoints of adjacent sides (as shown in the figure), then the area of the smaller square so produced is one-fifth that of the given square.





Areas and Perimeters of Regular Polygons

The area of a regular $2n$-gon inscribed in a circle is equal to one-half the radius of the circle times the perimeter of a regular n-gon similarly inscribed ($n \geq 3$).



$$\frac{1}{2n}\text{area}(P_{2n}) = \frac{1}{2}\cdot r \cdot \frac{1}{2}s_n$$

$$\therefore \text{area}(P_{2n}) = \frac{r}{2}ns_n$$

$$= \frac{r}{2}\text{perimeter}(P_n)$$

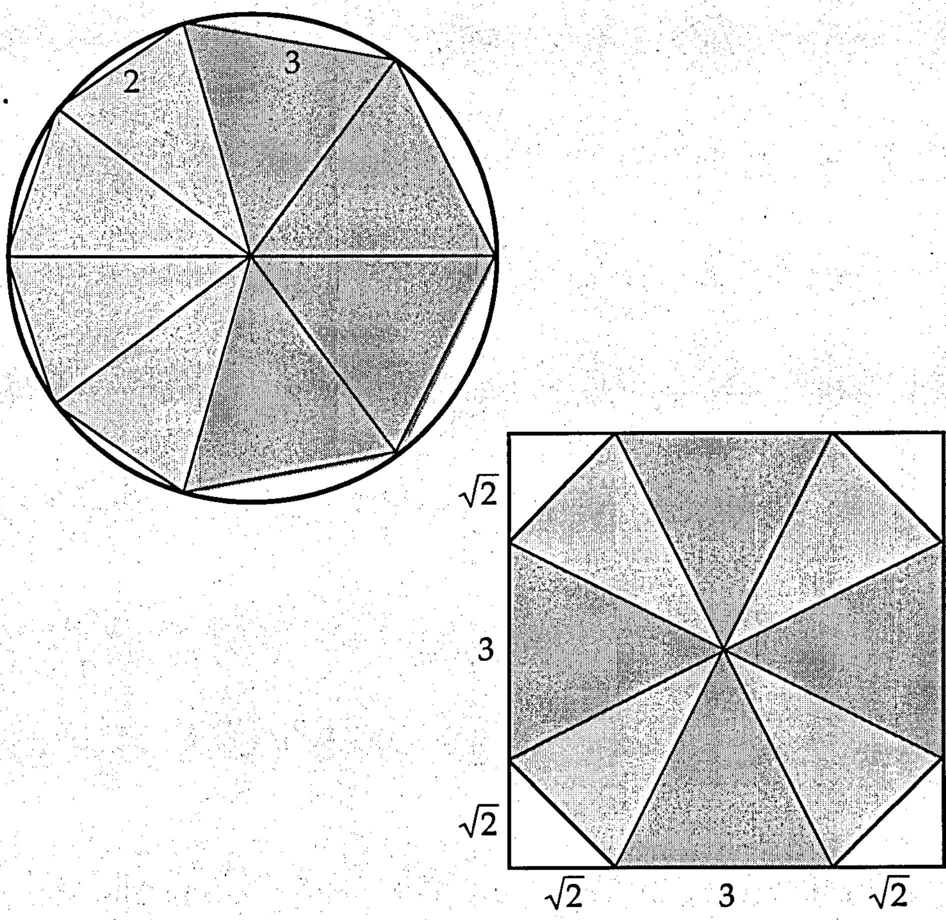
COROLLARY [Bhāskara, *Lilāvati* (India, 12th century A.D.)]: The area of a circle is equal to one-half the product of its radius and circumference.

The Area of a Putnam Octagon

(Problem B1, 39th Annual William Lowell Putnam Mathematical Competition, 1978)

Find the area of a convex octagon that is inscribed in a circle and has four consecutive sides of length 3 units and the remaining four sides of length 2 units. Give the answer in the form $r + s\sqrt{t}$ with r, s, and t positive integers.

SOLUTION:



$$A = \left(3 + 2\sqrt{2}\right)^2 - 4 \cdot \frac{1}{2}\left(\sqrt{2}\right)^2$$
$$= 13 + 12\sqrt{2}$$

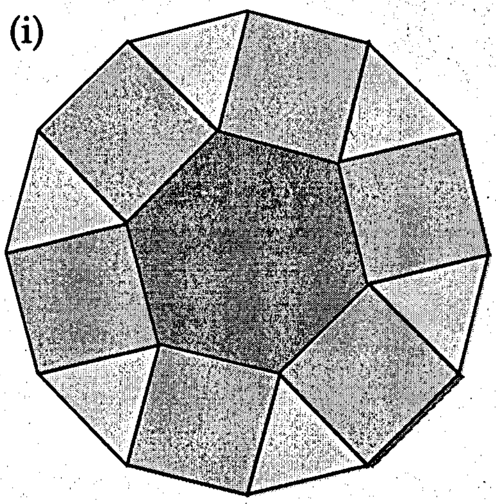
A Putnam Dodecagon

(Problem I-1, 24th Annual William Lowell Putnam Mathematical Competition, 1963)

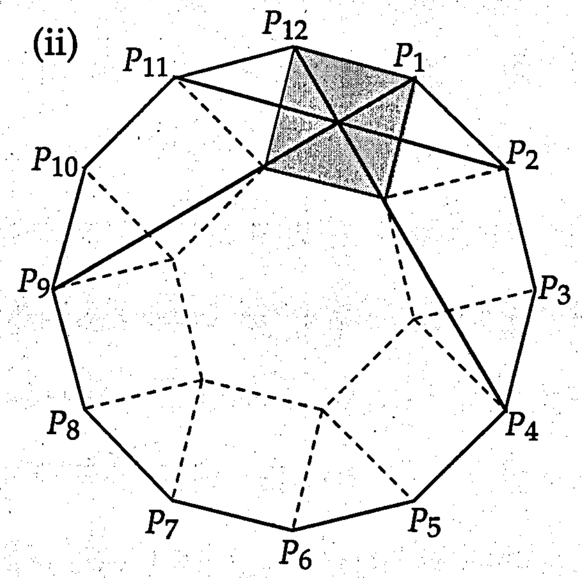
(i) Show that a regular hexagon, six squares, and six equilateral triangles can be assembled without overlapping to form a regular dodecagon.
(ii) Let $P_1, P_2, \ldots, P_{12}$ be the successive vertices of a regular dodecagon. Discuss the intersection(s) of the three diagonals P_1P_9, P_2P_{11}, and P_4P_{12}.

SOLUTION:

(i)



(ii)

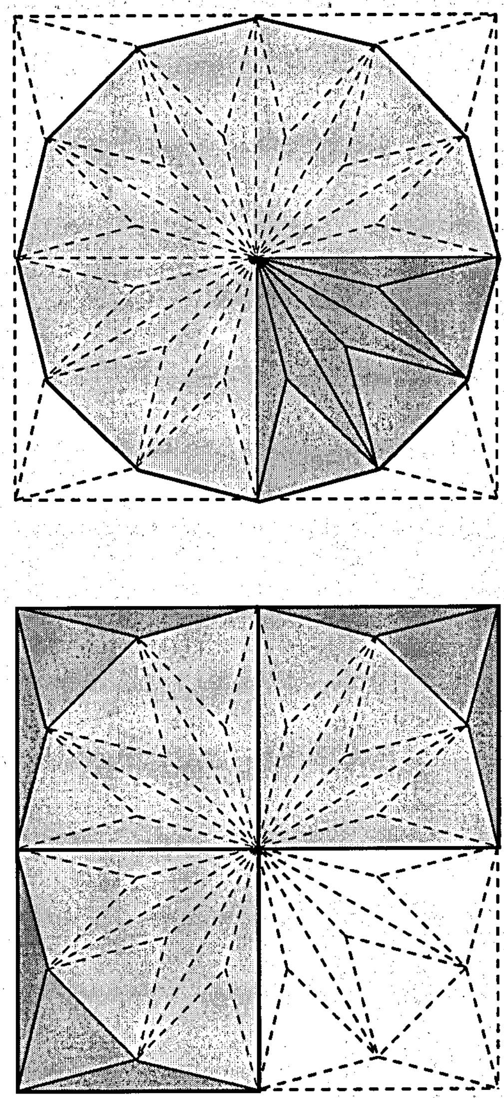


$\angle P_1P_{12}P_4 = \angle P_{12}P_1P_9 = \pi/4$,
$\angle P_1P_2P_{11} = \angle P_{12}P_{11}P_2 = \pi/6$,
$\therefore P_1P_9 \cap P_2P_{11} \cap P_4P_{12} \neq \varnothing$.

EXERCISE: Discuss the intersection(s) of the four diagonals P_1P_6, P_2P_9, P_3P_{11}, and P_4P_{12} (Problem F-4(b), *The AMATYC Review*, 1985, p. 61).

The Area of a Regular Dodecagon

A regular dodecagon with circumradius one has area three.

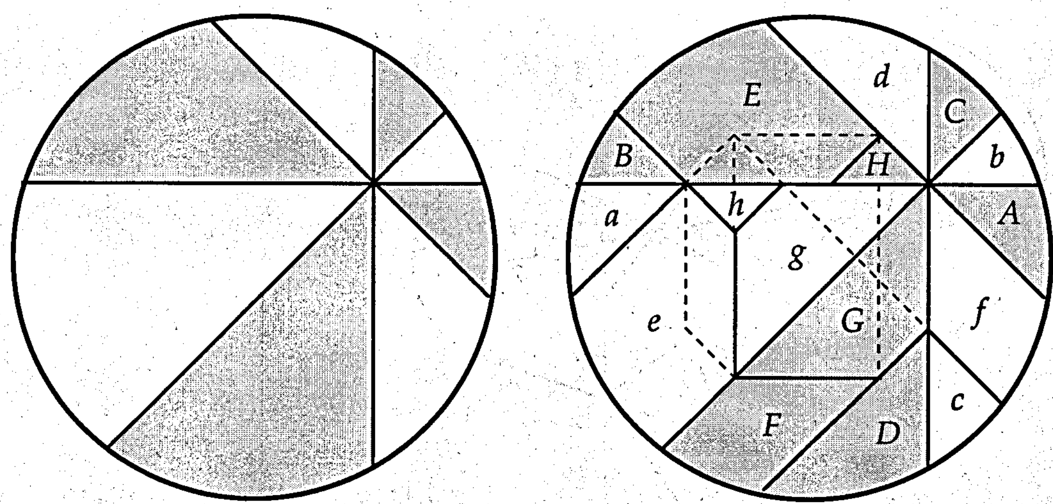


—J. Kürschák

Fair Allocation of a Pizza

THE PIZZA THEOREM: *If a pizza is divided into eight slices by making cuts at 45° angles from an arbitrary point in the pizza, then the sums of the areas of alternate slices are equal.*

PROOF:



NOTES: This result, discovered by L. J. Upton, is true when n, the number of pieces, is 8, 12, 16, ..., but false for $n = 2, 4, 6, 10, 14, 18, \ldots$. The positive results are in the references. For the negative, the case $n = 4$ is easily handled, while if $n \equiv 2 \pmod 4$ we have the following argument of Don Coppersmith (IBM). It suffices, by continuity, to take the special point on the boundary of the unit circle and one of the chords to be a tangent at the point. Then the gray area can be expressed in terms of π and algebraic numbers in such a way that its equality with $\pi/2$ would yield an algebraic relationship for π, in contradiction to π's transcendence (details omitted).

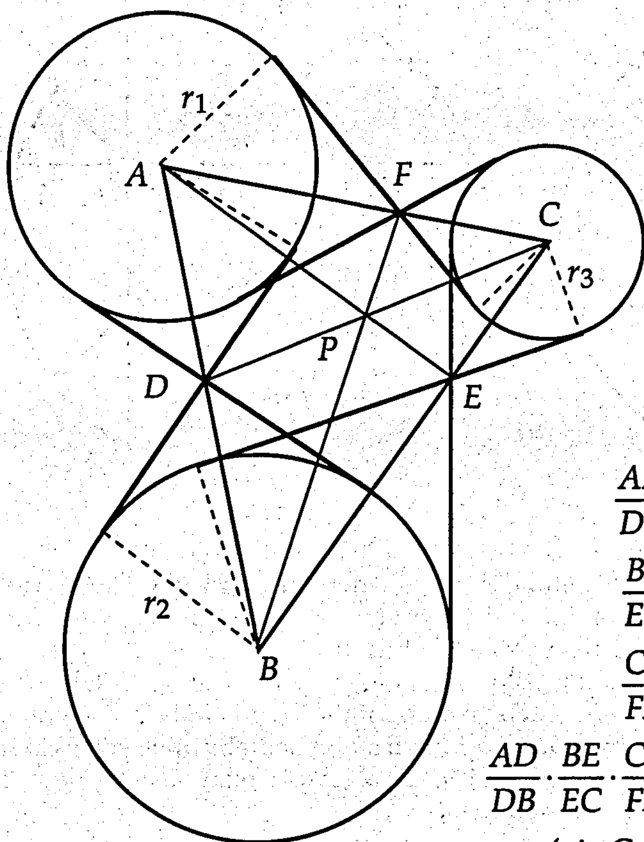
REFERENCES

1. L. J. Upton, Problem 660, *Mathematics Magazine* 41 (1968) 46.
2. S. Rabinowitz, Problem 1325, *Crux Mathematicorum* 15 (1989) 120-122.

—Larry Carter and Stan Wagon

A Three-Circle Theorem

Given three nonintersecting mutually external circles, connect the intersection of internal common tangents of each pair of circles with the center of the other circle. Then the resulting three line segments are concurrent.



$$\frac{AD}{DB}=\frac{r_1}{r_2}$$

$$\frac{BE}{EC}=\frac{r_2}{r_3}$$

$$\frac{CF}{FA}=\frac{r_3}{r_1}$$

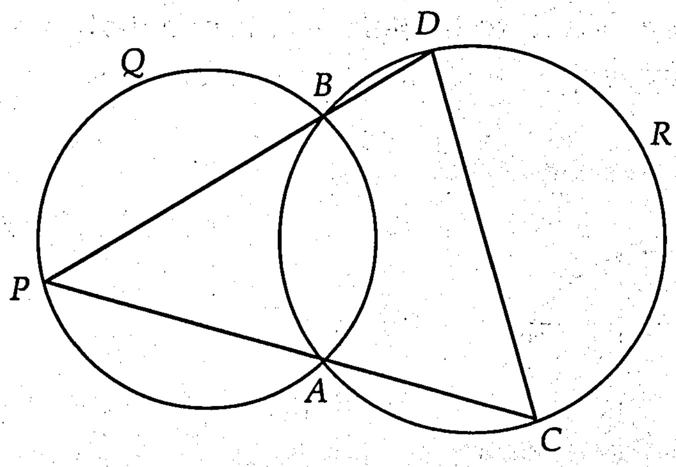
$$\frac{AD}{DB}\cdot\frac{BE}{EC}\cdot\frac{CF}{FA}=1 \Rightarrow AE\cap BF\cap CD=P$$

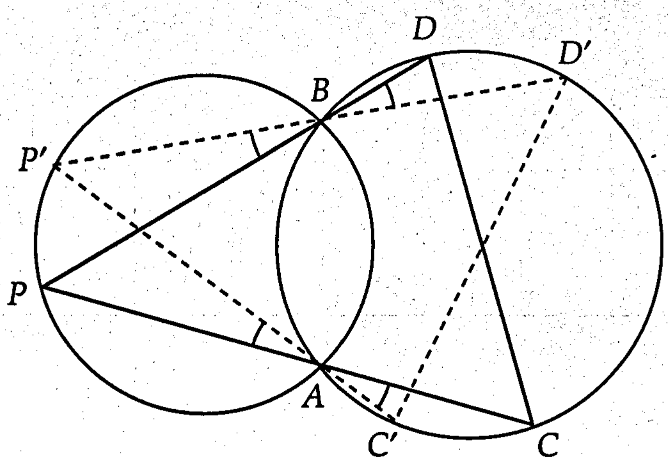
(via Ceva's theorem)

—R. S. Hu

A Constant Chord

Suppose two circles Q and R intersect in A and B. A point P on the arc of Q which lies outside R is projected through A and B to determine chord CD of R. Prove that no matter where P is chosen on its arc, the length of the chord CD is always the same.





$$\angle C'AC = \angle P'AP = \angle P'BP = \angle D'BD$$

$$\overset{\frown}{C'C} = \overset{\frown}{D'D}, \quad \overset{\frown}{C'D'} = \overset{\frown}{CD}$$

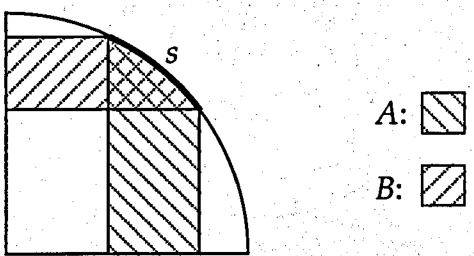
$$C'D' = CD$$

A Putnam Area Problem

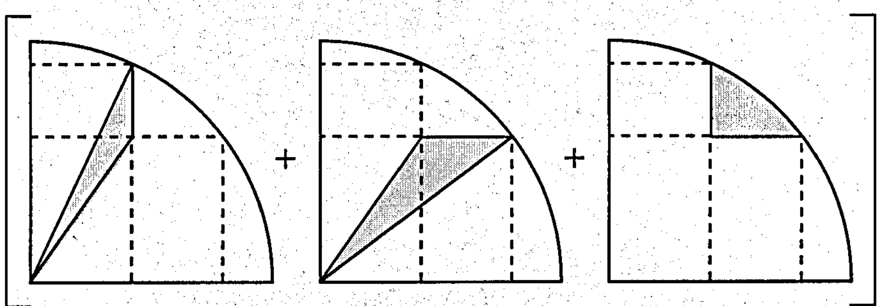
(Problem A2, 59th Annual William Lowell Putnam Mathematical Competition, 1998)

Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis, and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $A+B$ depends only on the arc length, and not the position, of s.

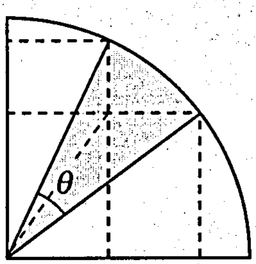
SOLUTION:



$= 2\times$



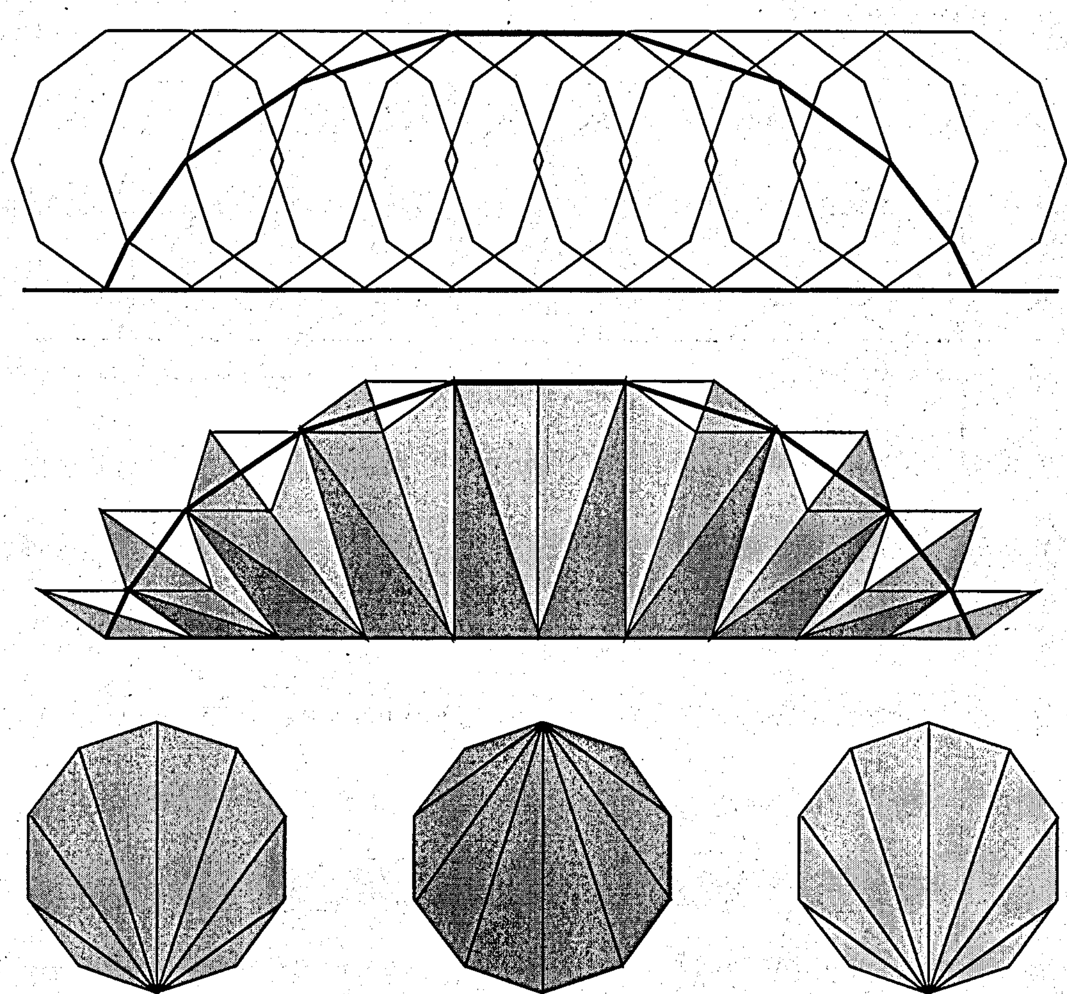
$= 2\times$



$$A+B = 2\times\frac{\theta}{2} = \theta = \ell(s)$$

The Area Under a Polygonal Arch

The area under the polygonal arch generated by one vertex of a regular n-gon rolling along a straight line is three times the area of the polygon.

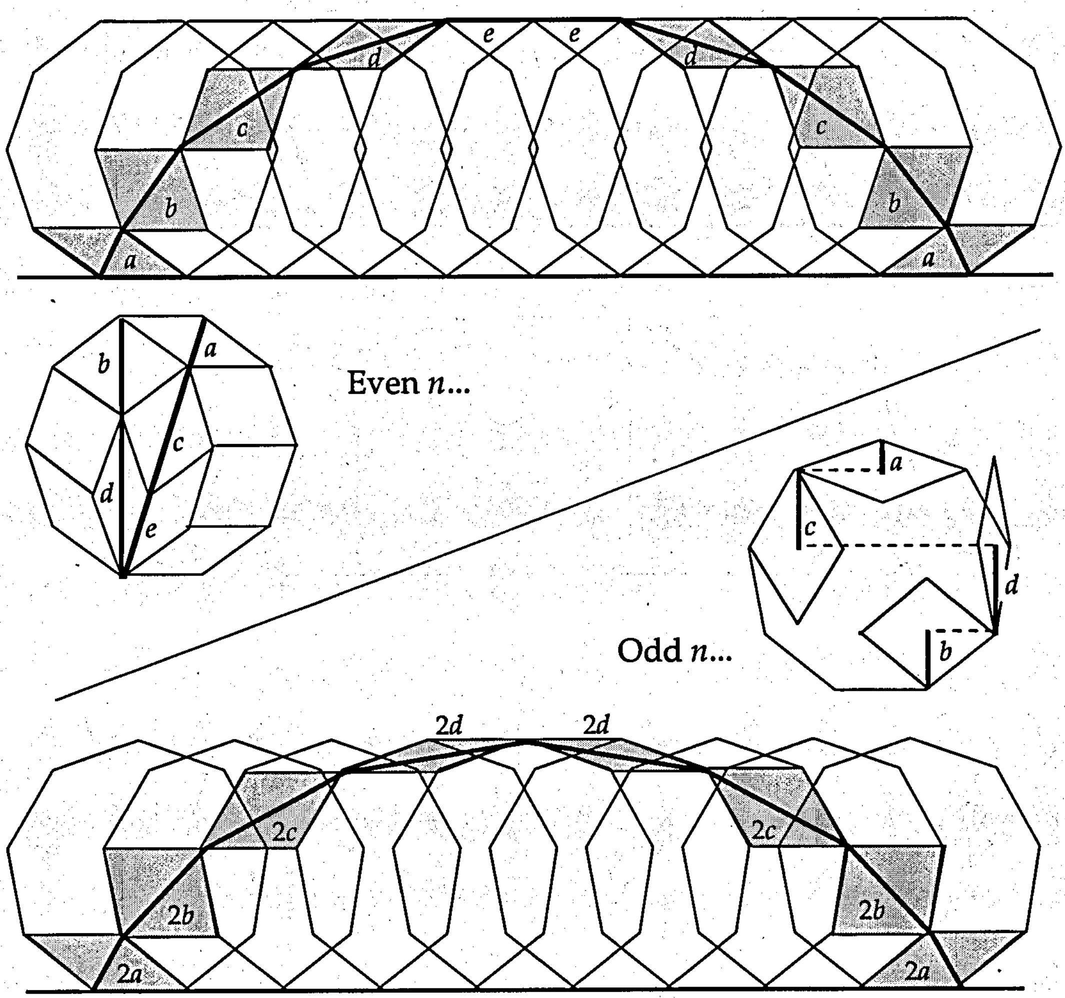


COROLLARY: The area under one arch of a cycloid is three times the area of the generating circle.

—Philip R. Mallinson

The Length of a Polygonal Arch

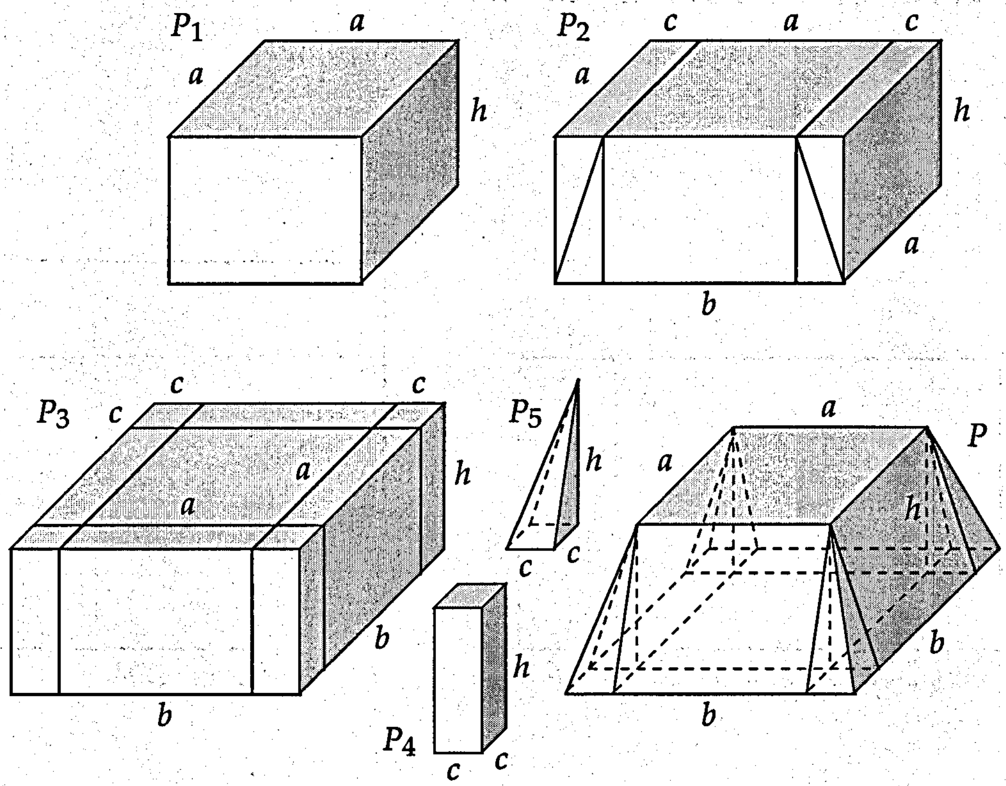
The length of the polygonal arch generated by one vertex of a regular n-gon rolling along a straight line is four times the length of the inradius plus four times the length of the circumradius of the n-gon.



COROLLARY: The arc length of one arch of a cycloid is eight times the radius of the generating circle.

— Philip R. Mallinson

The Volume of a Frustum of a Square Pyramid



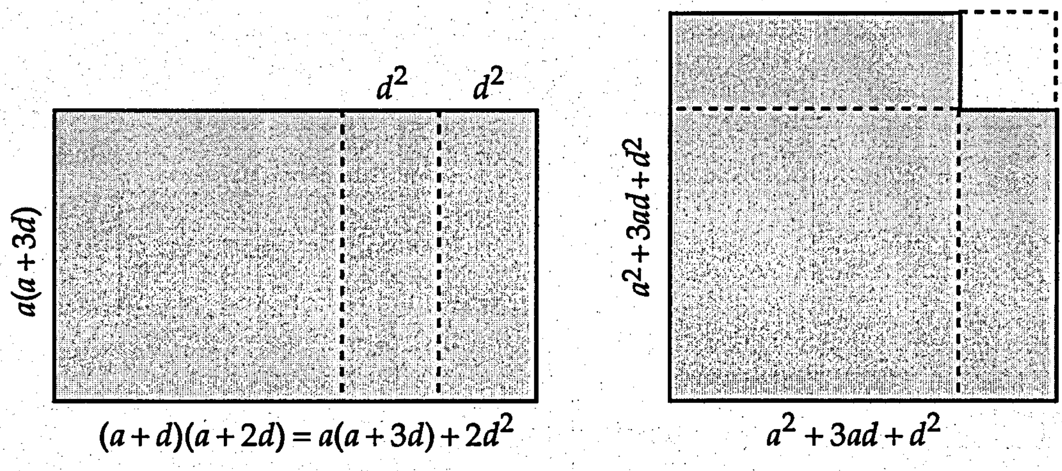
$$P_4 = 3P_5$$

$$P_1 + P_3 = 2P_2 + 4P_4 \Rightarrow P_1 + P_2 + P_3 = 3P_2 + 12P_5$$
$$= 3(P_2 + 4P_5) = 3P$$

$$\therefore V = \frac{h}{3}\left(a^2 + ab + b^2\right)$$

—Sidney J. Kung

The Product of Four (Positive) Numbers in Arithmetic Progression is Always the Difference of Two Squares

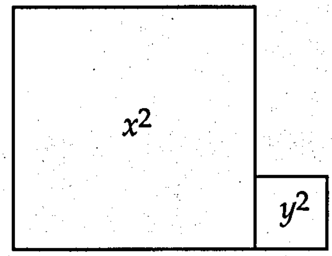


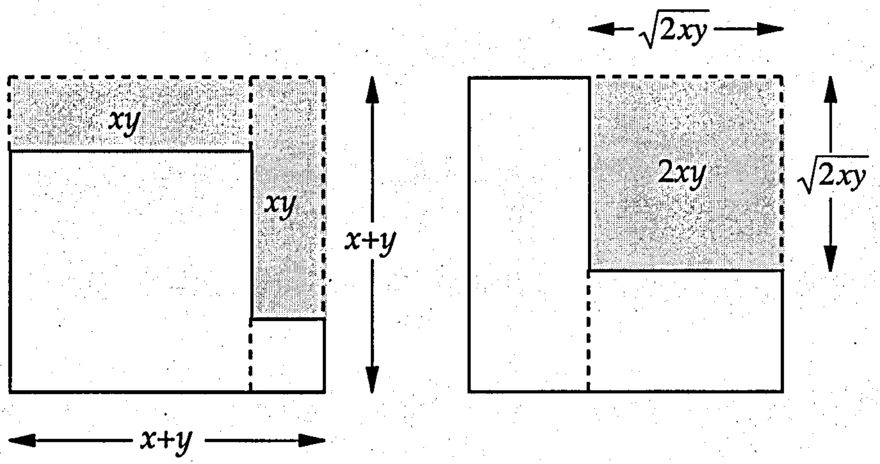
$$a(a+d)(a+2d)(a+3d)=(a^2+3ad+d^2)^2-(d^2)^2$$

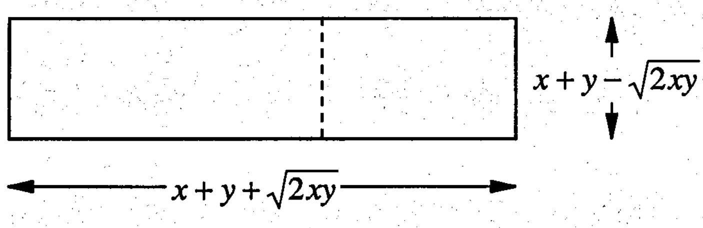
—RBN

Algebraic Areas III: Factoring the Sum of Two Squares

$$x^2 + y^2 = \left(x + \sqrt{2xy} + y\right)\left(x - \sqrt{2xy} + y\right)$$

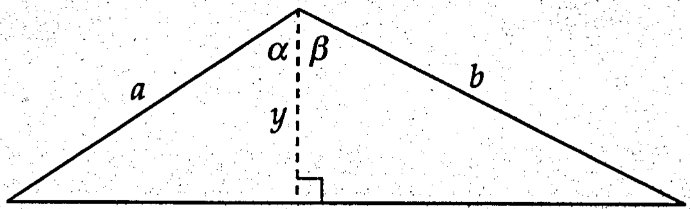




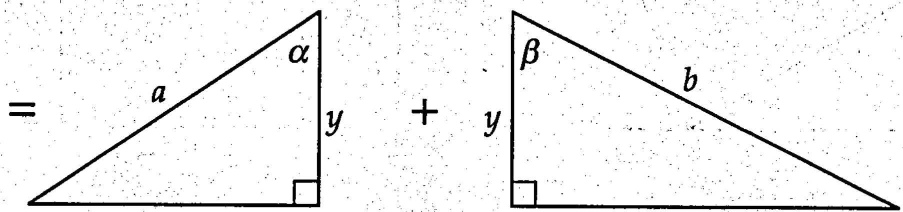


Trigonometry, Calculus, & Analytic Geometry

Sine of the Sum II



$$\alpha, \beta \in (0, \pi/2) \Rightarrow y = a\cos\alpha = b\cos\beta$$



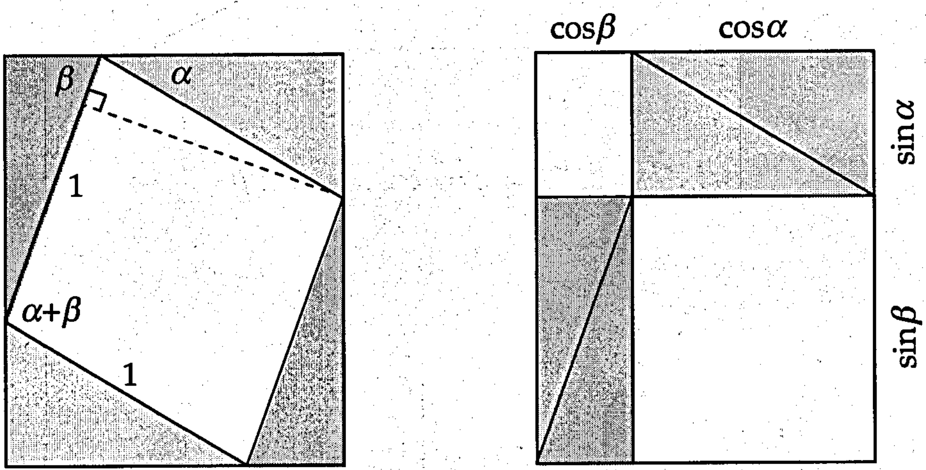
$$\frac{1}{2}ab\sin(\alpha+\beta) = \frac{1}{2}ay\sin\alpha + \frac{1}{2}by\sin\beta$$
$$= \frac{1}{2}ab\cos\beta\sin\alpha + \frac{1}{2}ba\cos\alpha\sin\beta$$
$$\therefore \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

—Christopher Brueningsen

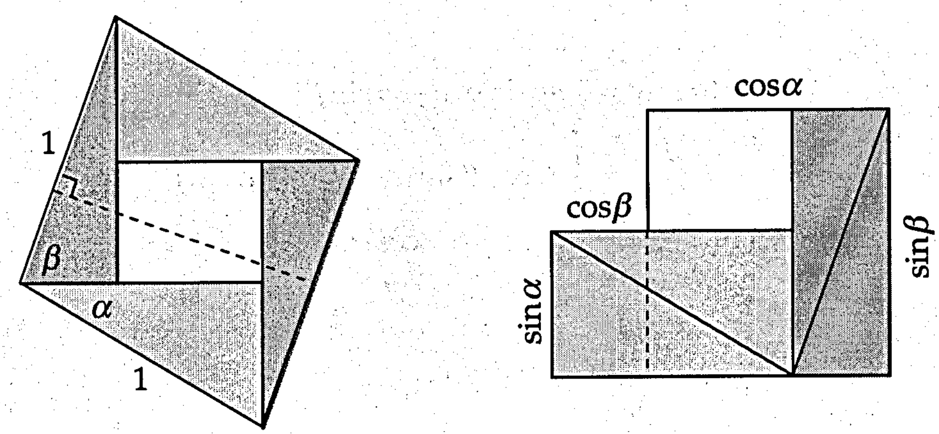
Sine of the Sum III

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\sin\beta\cos\alpha$$

I.

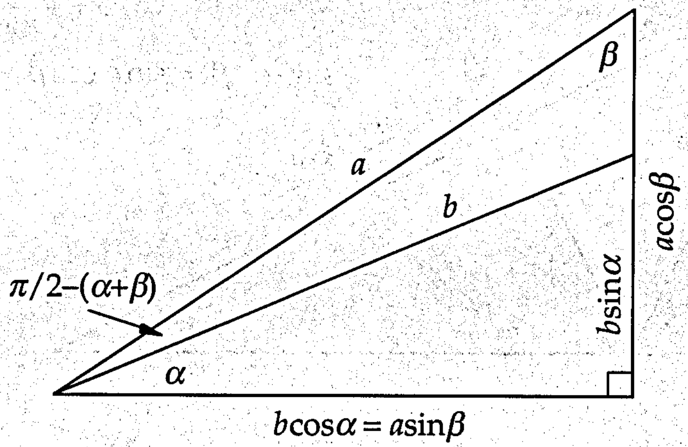


II.



—Volker Priebe
and Edgar A. Ramos

Cosine of the Sum

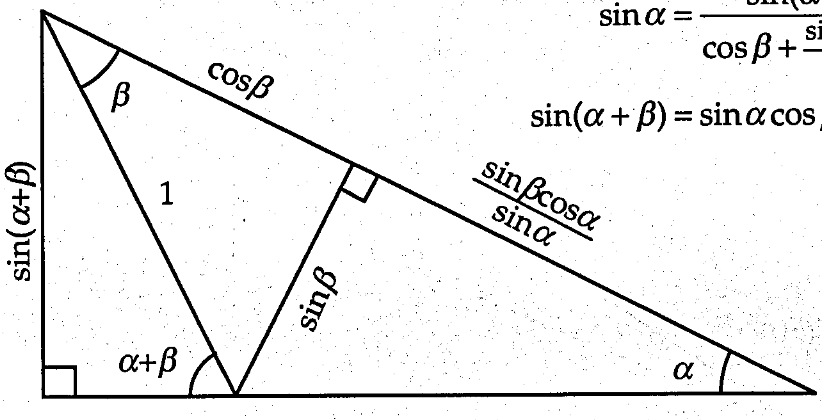


$$\frac{1}{2}ab\sin\left[\frac{\pi}{2}-(\alpha+\beta)\right]=\frac{1}{2}b\cos\alpha\cdot a\cos\beta-\frac{1}{2}a\sin\beta\cdot b\sin\alpha$$

$$\therefore \cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

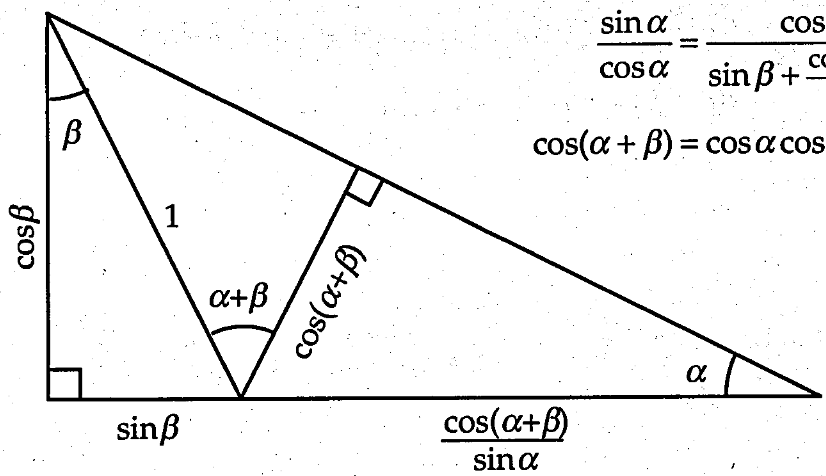
—Sidney H. Kung

Geometry of Addition Formulas



$$\sin\alpha = \frac{\sin(\alpha+\beta)}{\cos\beta + \frac{\sin\beta\cos\alpha}{\sin\alpha}}$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

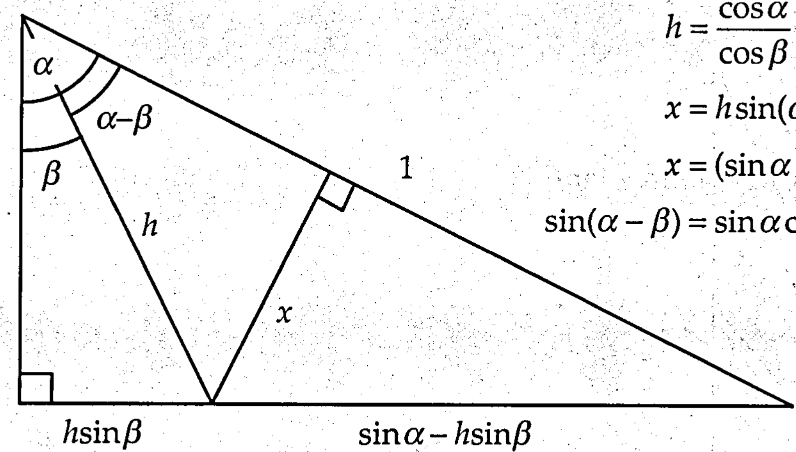


$$\frac{\sin\alpha}{\cos\alpha} = \frac{\cos\beta}{\sin\beta + \frac{\cos(\alpha+\beta)}{\sin\alpha}}$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

—Leonard M. Smiley

Geometry of Subtraction Formulas

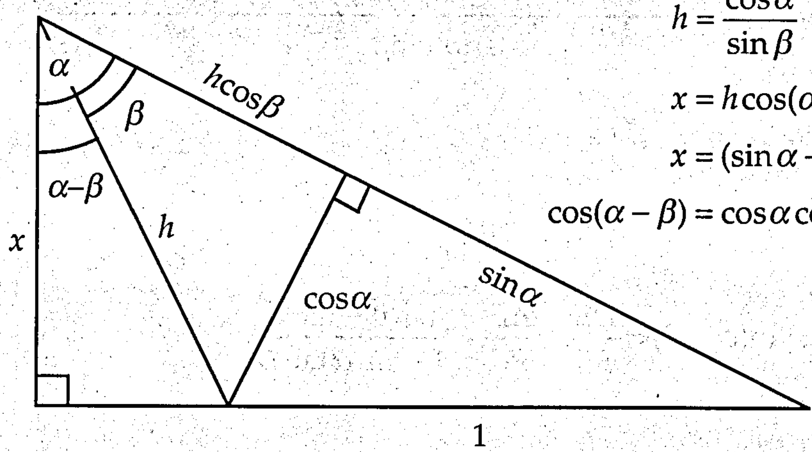


$$h = \frac{\cos\alpha}{\cos\beta}$$

$$x = h\sin(\alpha - \beta)$$

$$x = (\sin\alpha - h\sin\beta)\cos\alpha$$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$



$$h = \frac{\cos\alpha}{\sin\beta}$$

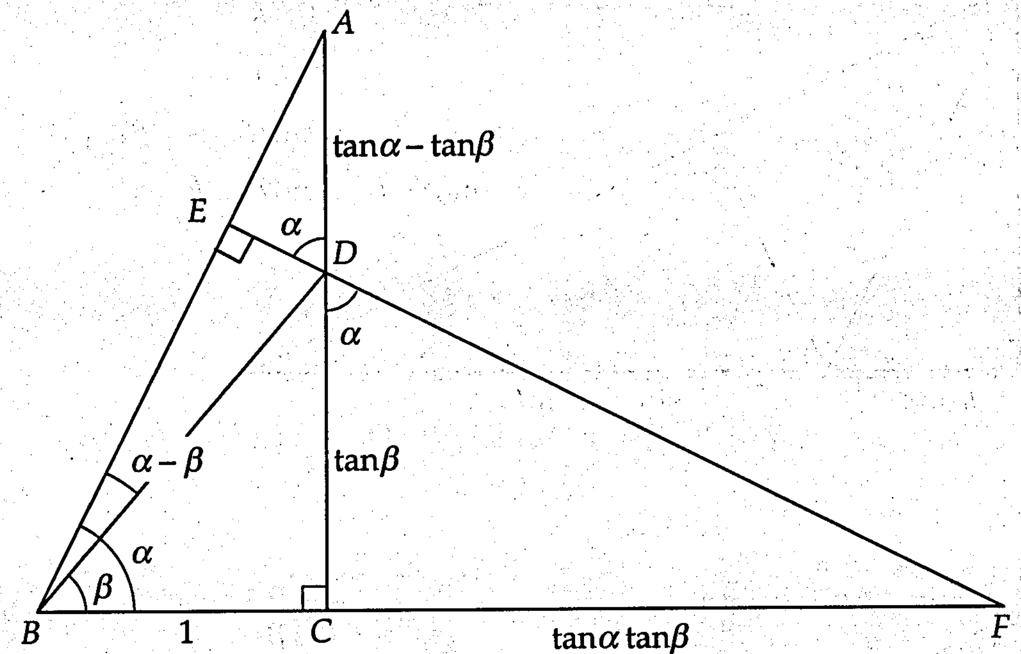
$$x = h\cos(\alpha - \beta)$$

$$x = (\sin\alpha - h\cos\beta)\cos\alpha$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

—Leonard M. Smiley

The Difference Identity for Tangents I

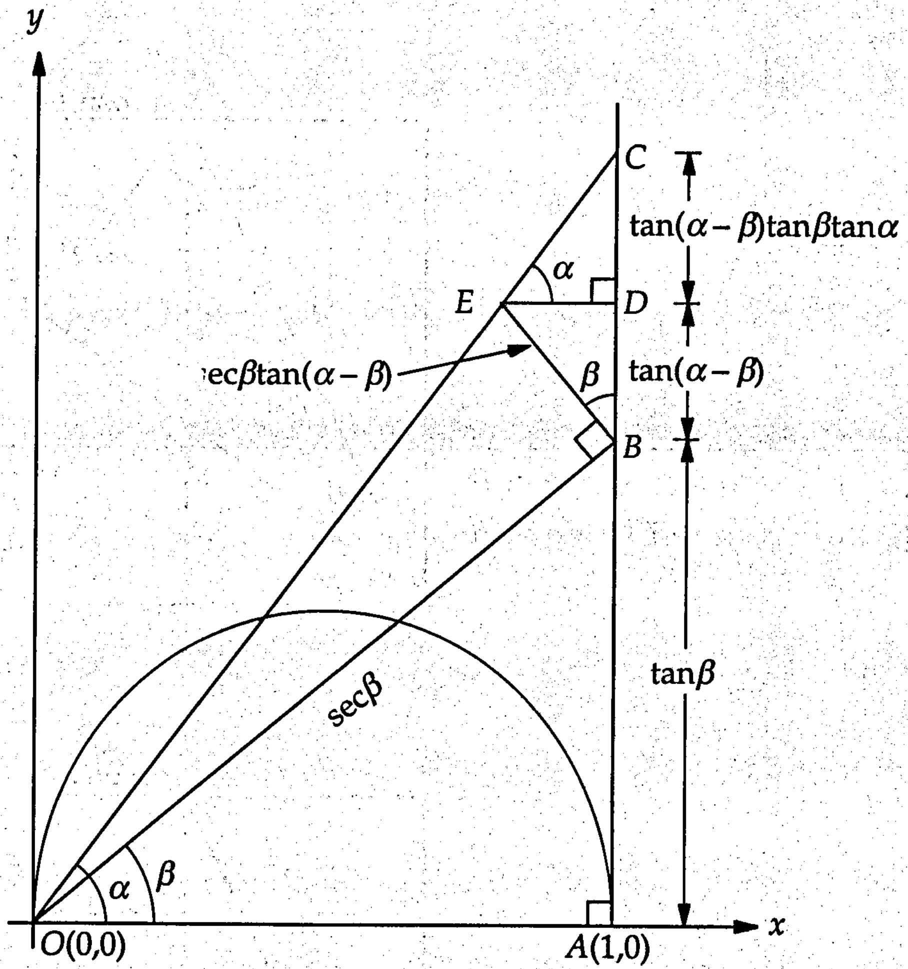


$$\frac{BF}{BE} = \frac{AD}{DE},$$

$$\therefore \tan(\alpha - \beta) = \frac{DE}{BE} = \frac{AD}{BF} = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.$$

—Guanshen Ren

The Difference Identity for Tangents II



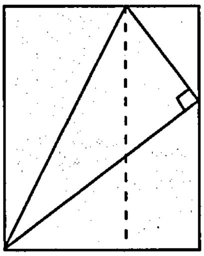
$$AC - AB = BD + DC$$

$$\therefore \tan\alpha - \tan\beta = \tan(\alpha-\beta) + \tan\alpha\tan\beta\tan(\alpha-\beta)$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}.$$

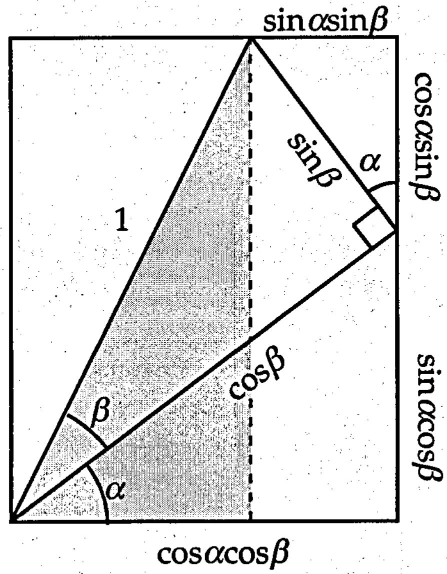
—Fukuzo Suzuki

One Figure, Six Identities

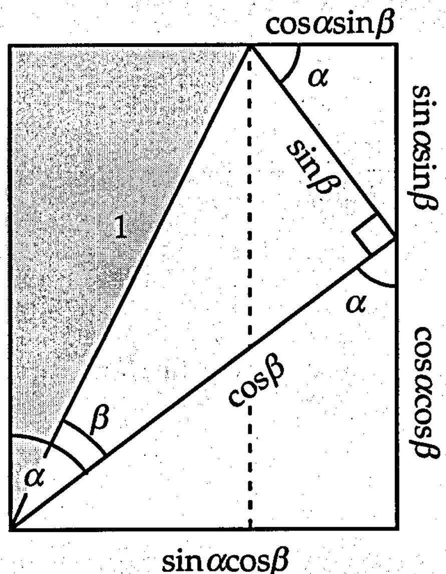


The figure

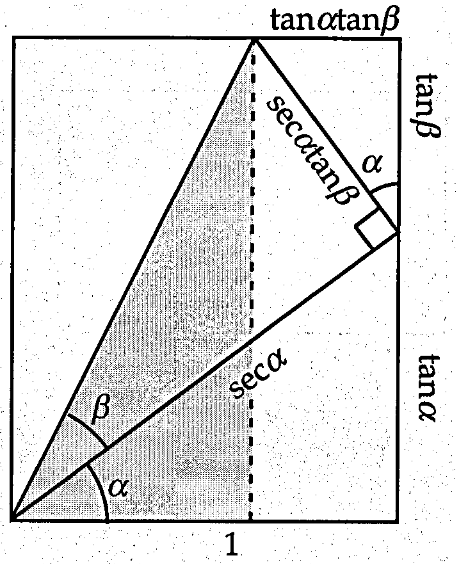
$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$



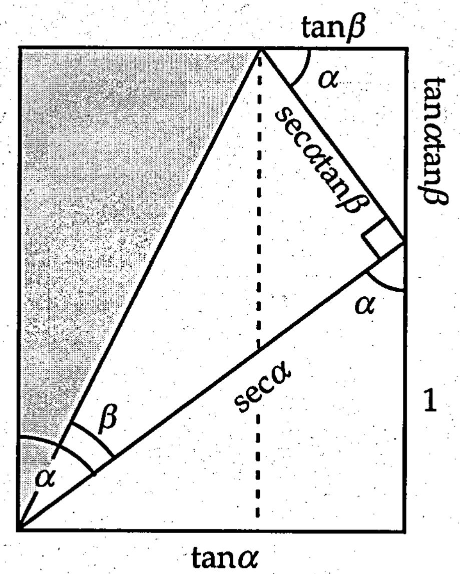
$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$



$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

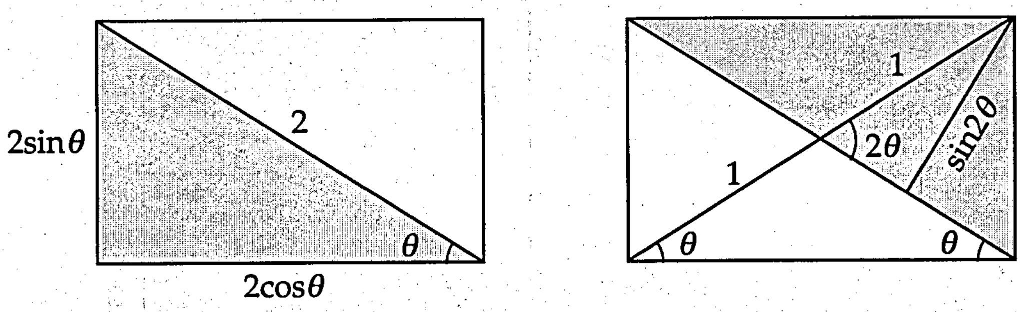


$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

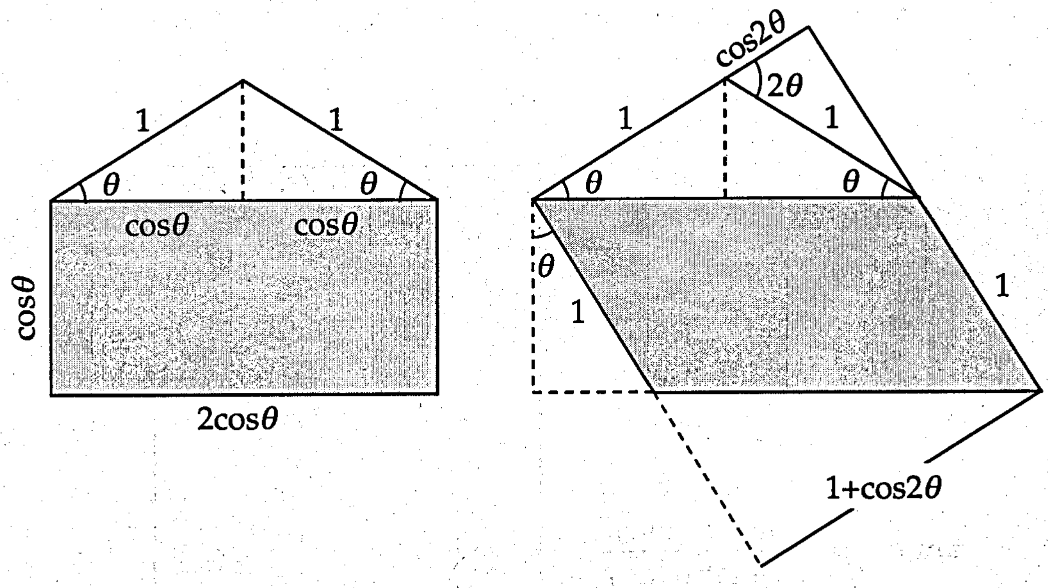


—RBN

The Double-Angle Formulas II



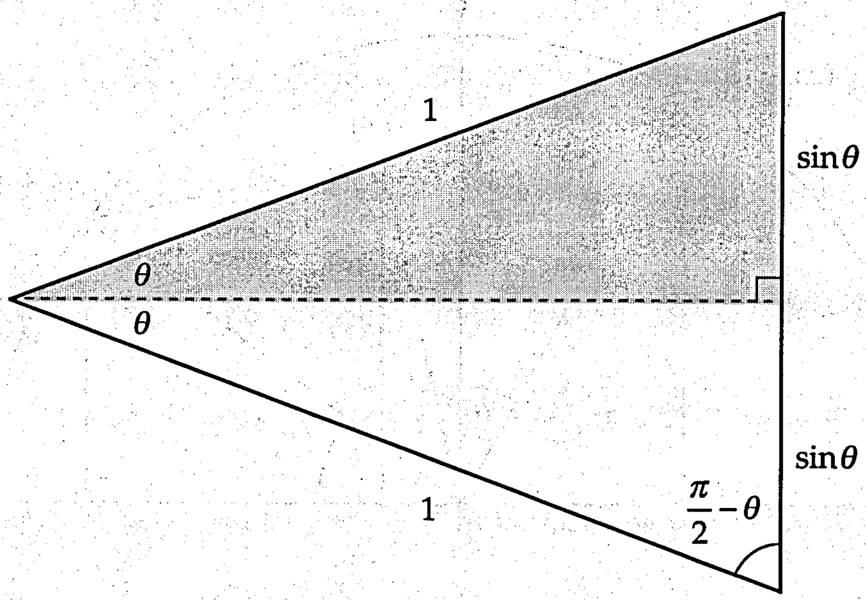
$$2\sin\theta\cos\theta = \sin 2\theta$$



$$2\cos^2\theta = 1 + \cos 2\theta$$

—Yihnan David Gau

The Double-Angle Formulas III (via the Laws of Sines and Cosines)

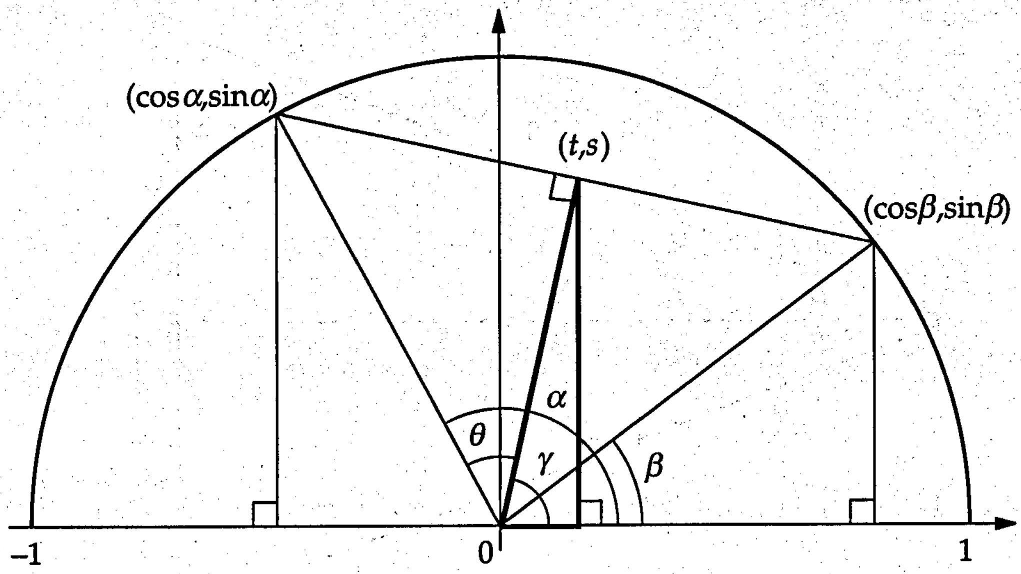


$$\frac{\sin 2\theta}{2\sin\theta} = \frac{\sin(\pi/2-\theta)}{1} = \cos\theta$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$(2\sin\theta)^2 = 1^2 + 1^2 - 2\cdot 1\cdot 1\cdot\cos 2\theta$$
$$\cos 2\theta = 1 - 2\sin^2\theta$$

—Sidney H. Kung

The Sum-to-Product Identities I



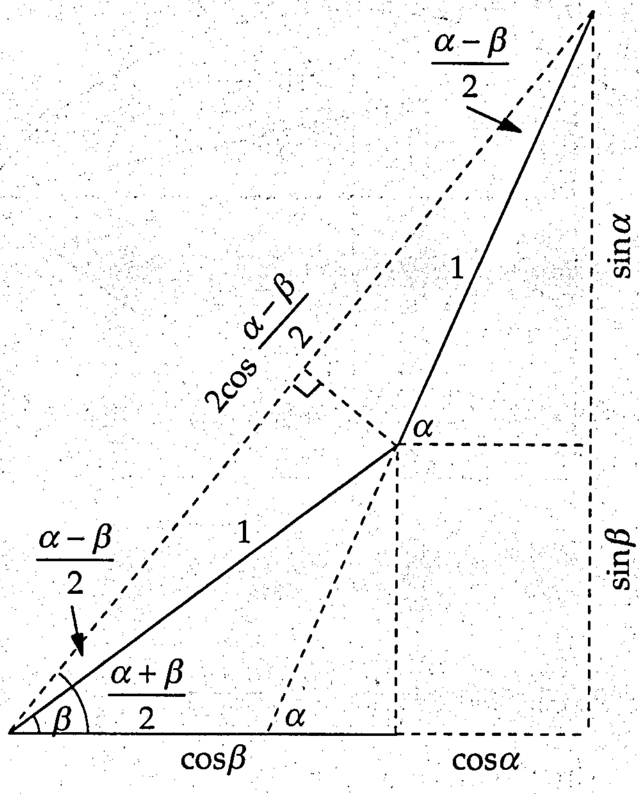
$$\theta = \frac{\alpha-\beta}{2}, \quad \gamma = \frac{\alpha+\beta}{2}$$

$$\frac{\sin\alpha+\sin\beta}{2} = s = \cos\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}$$

$$\frac{\cos\alpha+\cos\beta}{2} = t = \cos\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$

—Sidney H. Kung

The Sum-to-Product Identities II

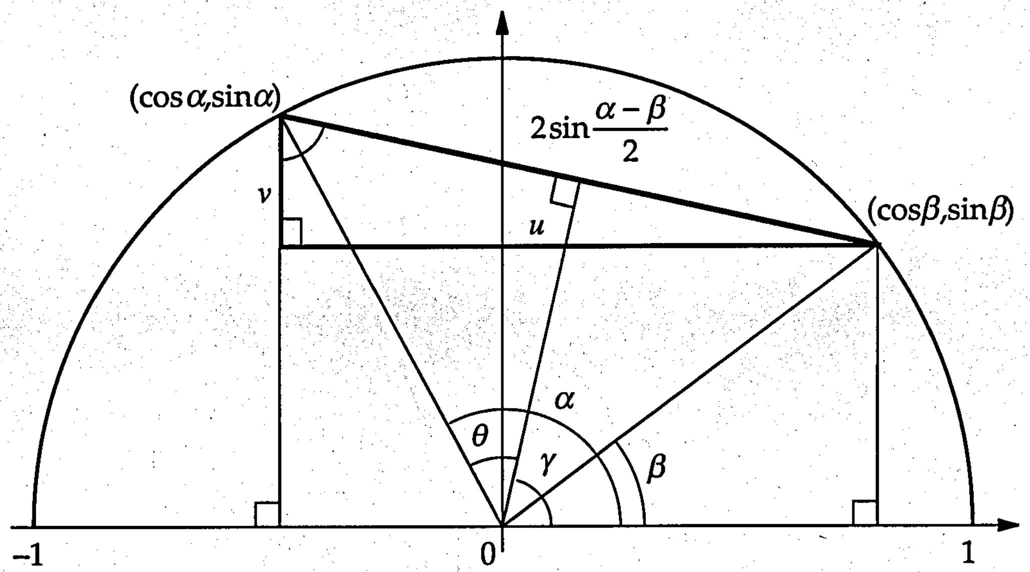


$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$

$$\sin\alpha + \sin\beta = 2\cos\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}$$

—Yukio Kobayashi

The Difference-to-Product Identities I



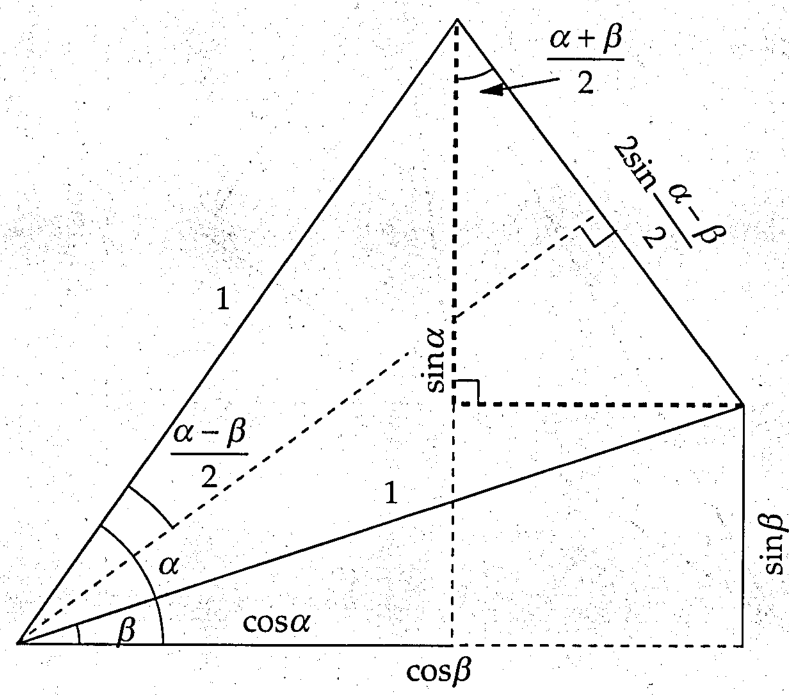
$$\theta = \frac{\alpha - \beta}{2}, \quad \gamma = \frac{\alpha + \beta}{2}$$

$$\sin\alpha - \sin\beta = v = 2\sin\frac{\alpha - \beta}{2}\cos\frac{\alpha + \beta}{2}$$

$$\cos\beta - \cos\alpha = u = 2\sin\frac{\alpha - \beta}{2}\sin\frac{\alpha + \beta}{2}$$

— Sidney H. Kung

The Difference-to-Product Identities II

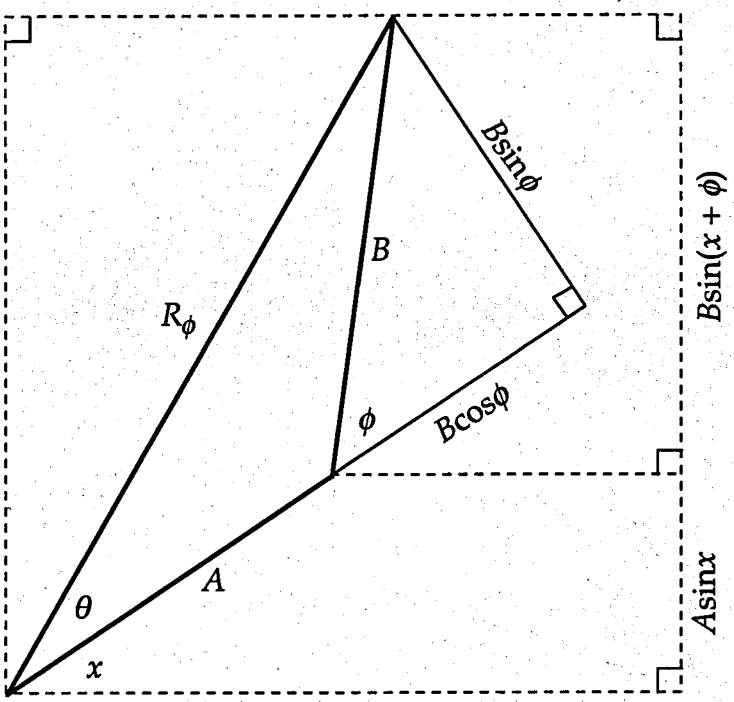


$$\cos\beta - \cos\alpha = 2\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}$$

$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$

— Yukio Kobayashi

Adding Like Sines



$$R_\phi = \sqrt{A^2 + B^2 + 2AB\cos\phi}, \quad \tan\theta = \frac{B\sin\phi}{A + B\cos\phi}$$

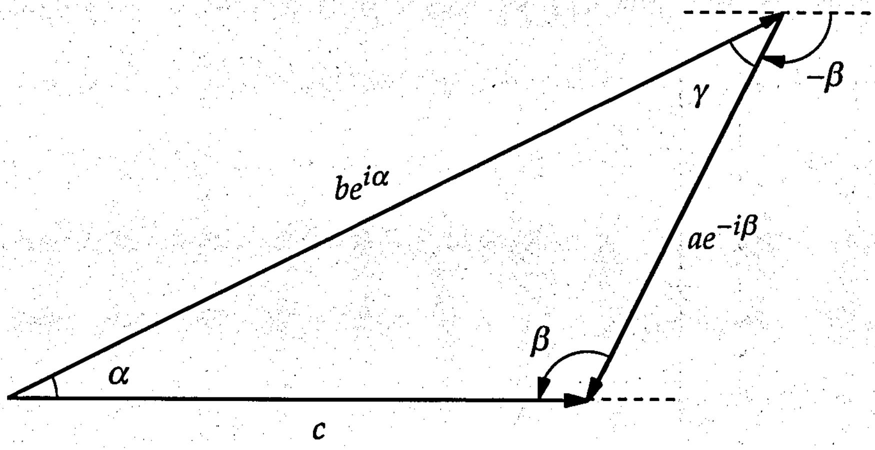
$$A\sin x + B\sin(x + \phi) = R_\phi \sin(x + \theta)$$

$$\phi = \frac{\pi}{2} \Rightarrow \tan\theta = \frac{B}{A}$$

$$\therefore A\sin x + B\cos x = \sqrt{A^2 + B^2}\sin(x + \theta)$$

—Rick Mabry
and Paul Deiermann

A Complex Approach to the Laws of Sines and Cosines



$$c = be^{i\alpha} + ae^{-i\beta} = (b\cos\alpha + a\cos\beta) + i(b\sin\alpha - a\sin\beta)$$

$$c \text{ real} \Rightarrow b\sin\alpha - a\sin\beta = 0 \Rightarrow \frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

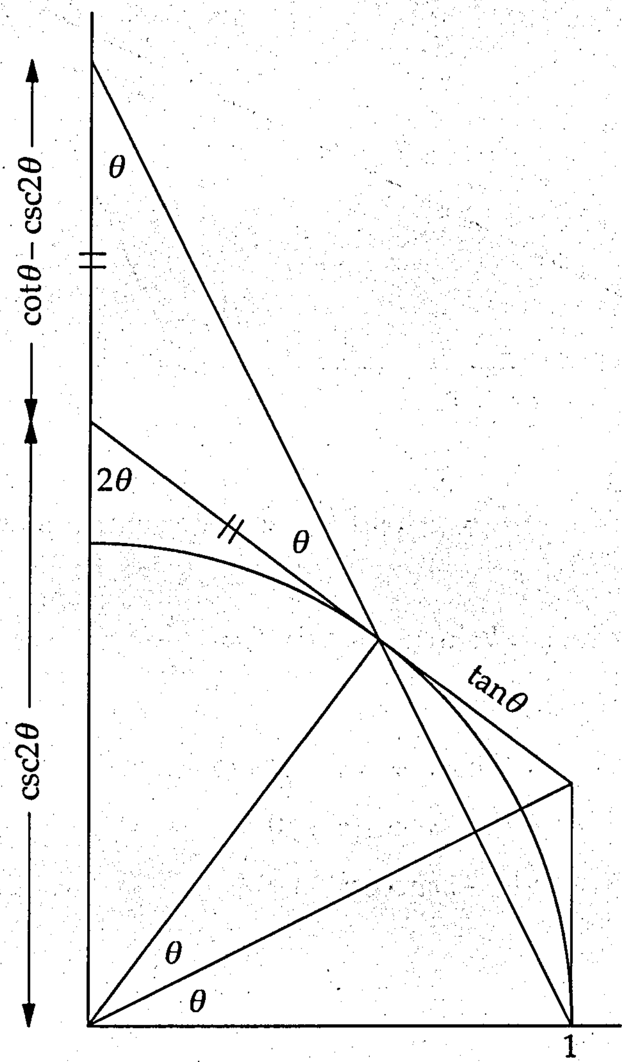
$$\begin{aligned} c^2 = |c|^2 &= (b\cos\alpha + a\cos\beta)^2 + (b\sin\alpha - a\sin\beta)^2 \\ &= a^2 + b^2 + 2ab\cos(\alpha + \beta) \\ &= a^2 + b^2 - 2ab\cos\gamma \end{aligned}$$

—William V. Grounds

Eisenstein's Duplication Formula

(G. Eisenstein, *Mathematische Werke*, Chelsea, New York, 1975, p. 411)

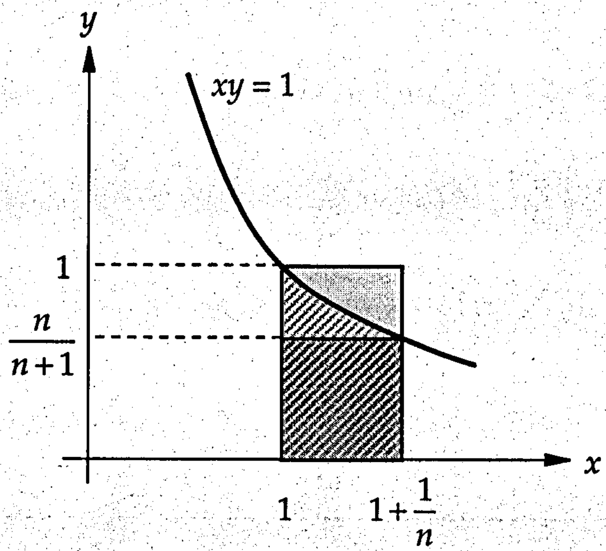
$$2\csc 2\theta = \tan\theta + \cot\theta$$



—Lin Tan

A Familiar Limit for e

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$



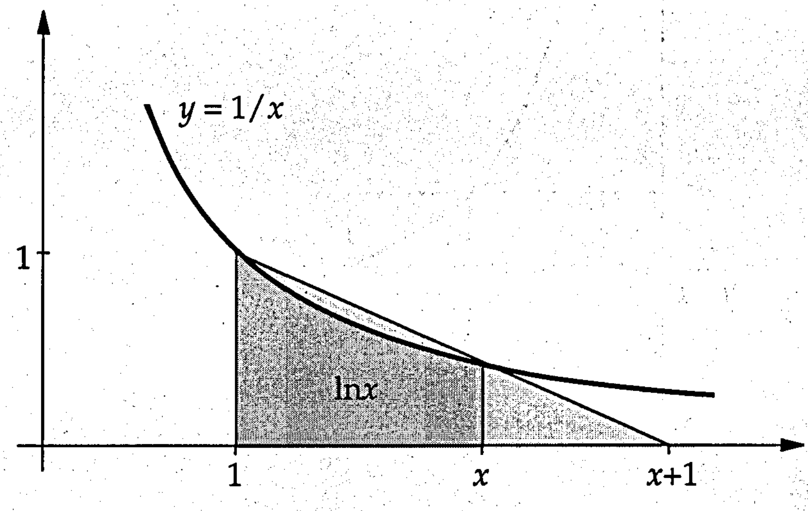
$$\frac{1}{n}\cdot\frac{n}{n+1} \le \ln\left(1+\frac{1}{n}\right) \le \frac{1}{n}\cdot 1$$

$$\frac{n}{n+1} \le n\cdot\ln\left(1+\frac{1}{n}\right) \le 1$$

$$\therefore \lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n = 1$$

A Common Limit

$$\lim_{x\to\infty} \frac{x}{e^x} = 0$$



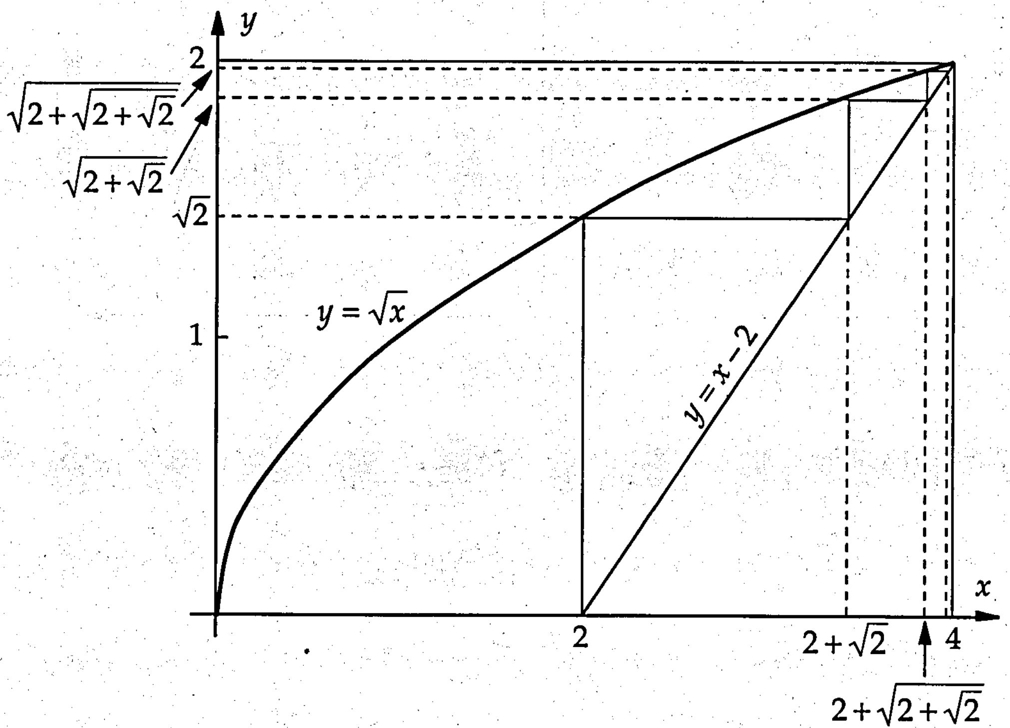
$$\ln x < \frac{1}{2}x$$

$$\therefore \lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^{x-\ln x}} = 0$$

—Alan H. Stein
and Dennis McGavran

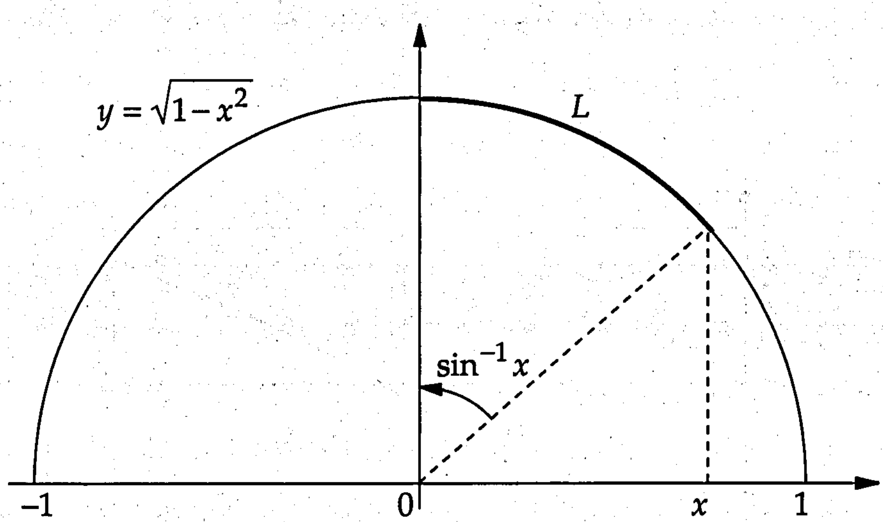
Geometric Evaluation of a Limit

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{\cdots}}}}=2$$



—Guanshen Ren

The Derivative of the Inverse Sine



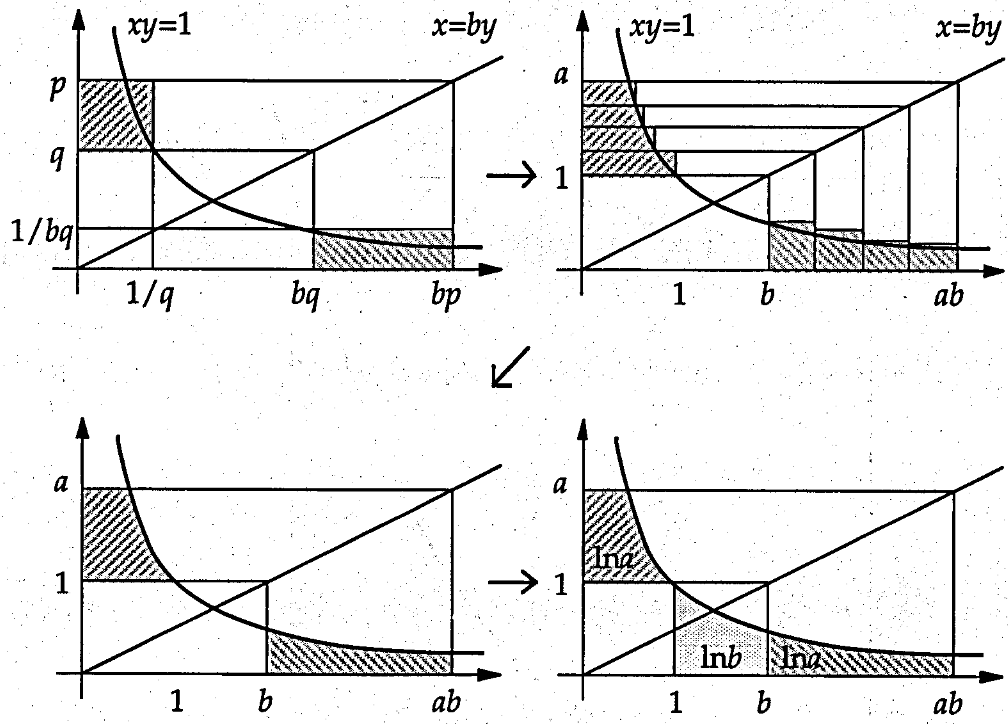
$$L = \sin^{-1}x = \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt$$

$$\therefore \frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$

—Craig Johnson

The Logarithm of a Product

$$\ln ab = \ln a + \ln b$$

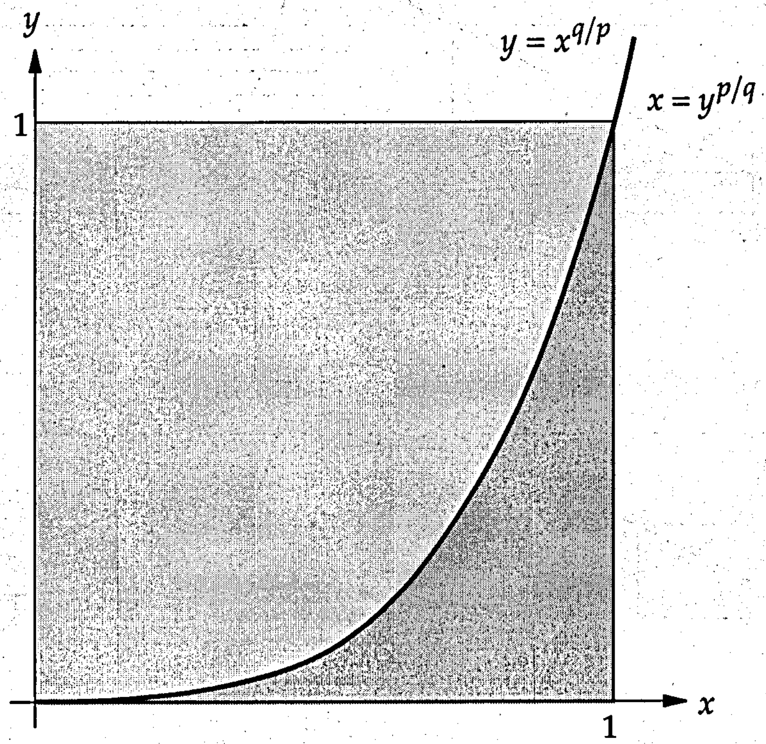


Area(▨) = Area(▧)

—Jeffrey Ely

An Integral of a Sum of Reciprocal Powers

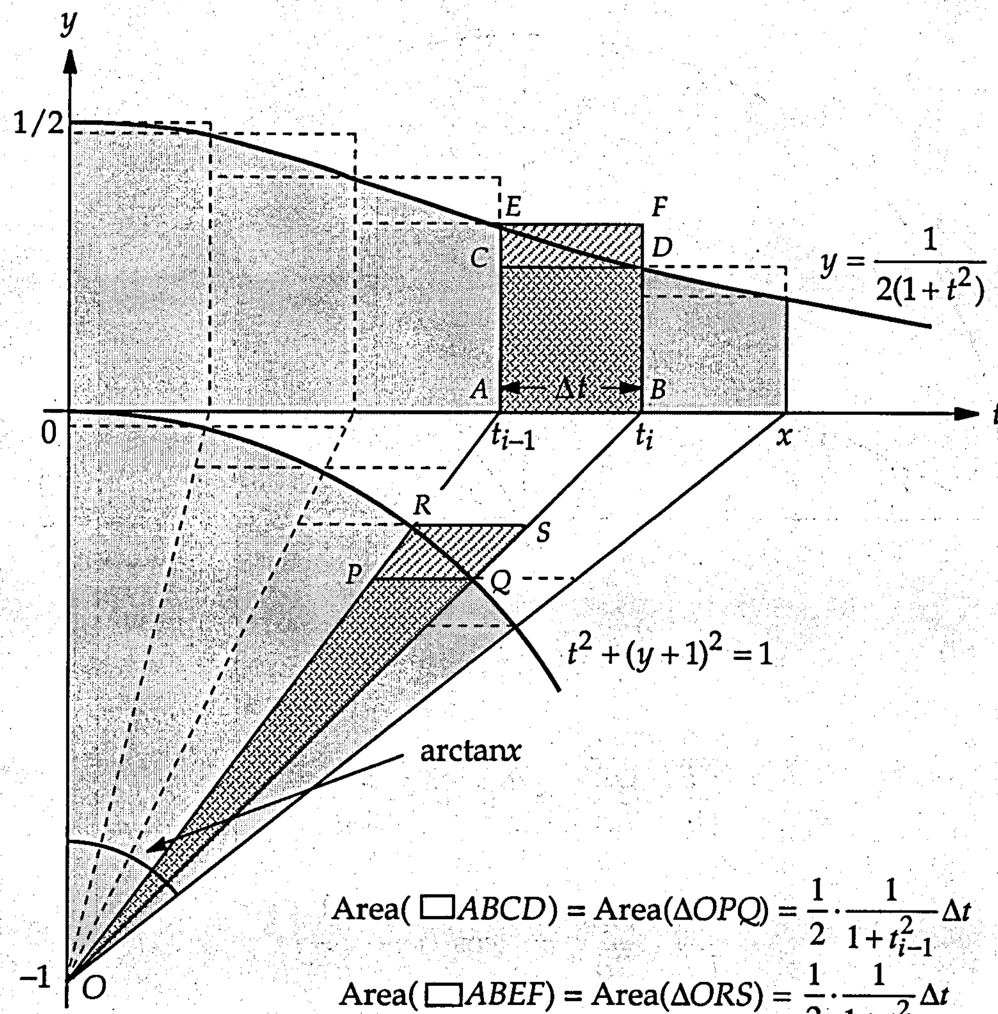
$$\int_0^1 \left(t^{p/q} + t^{q/p}\right) dt = 1$$



—Peter R. Newbury

The Arctangent Integral

$$\arctan x = \int_0^x \frac{1}{1+t^2}\,dt$$



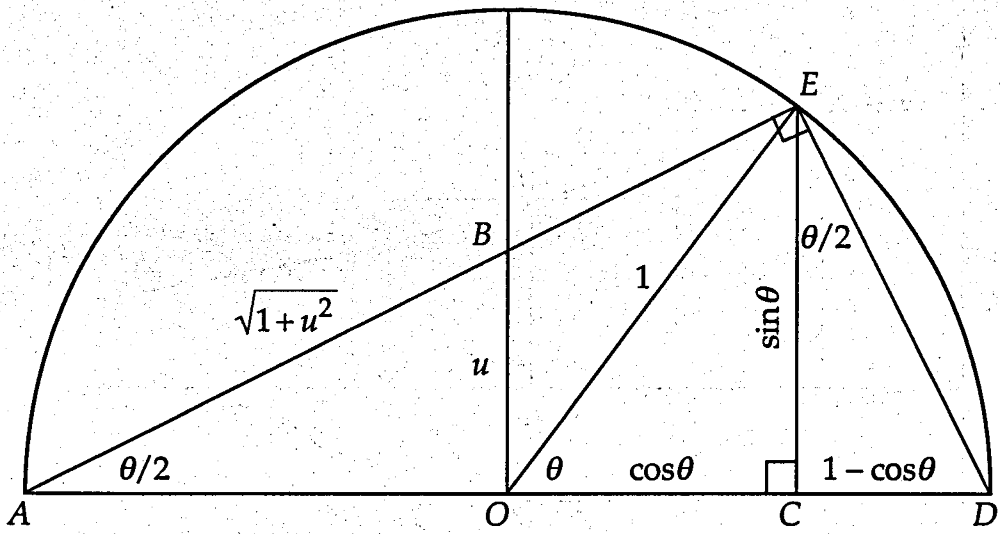
$$\text{Area}(\square ABCD) = \text{Area}(\Delta OPQ) = \frac{1}{2}\cdot\frac{1}{1+t_{i-1}^2}\Delta t$$

$$\text{Area}(\square ABEF) = \text{Area}(\Delta ORS) = \frac{1}{2}\cdot\frac{1}{1+t_i^2}\Delta t$$

$$\therefore \frac{1}{2}\arctan x = \int_0^x \frac{1}{2(1+t^2)}\,dt$$

—Aage Bondesen

The Method of Last Resort (Weierstrass Substitution)



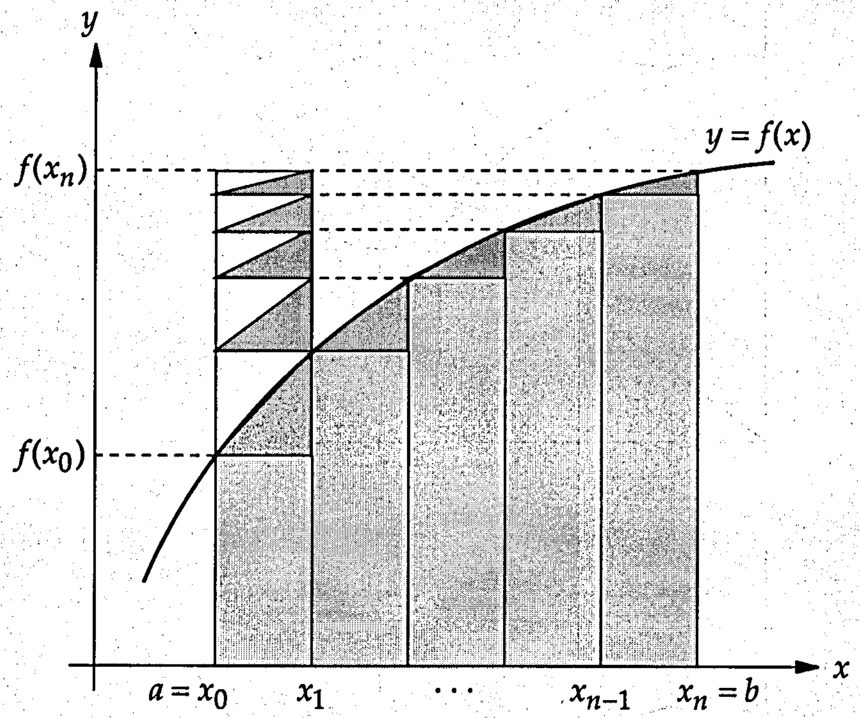
$$u = \tan\frac{\theta}{2}, \quad \overline{DE} = 2\sin\frac{\theta}{2} = \frac{2u}{\sqrt{1+u^2}}$$

$$\frac{\overline{CE}}{\overline{DE}} = \frac{\overline{OA}}{\overline{BA}} \quad \Rightarrow \quad \sin\theta = \frac{2u}{1+u^2}$$

$$\frac{\overline{CD}}{\overline{DE}} = \frac{\overline{OB}}{\overline{BA}} \quad \Rightarrow \quad \cos\theta = \frac{1-u^2}{1+u^2}$$

—Paul Deiermann

The Trapezoidal Rule (for Increasing Functions)

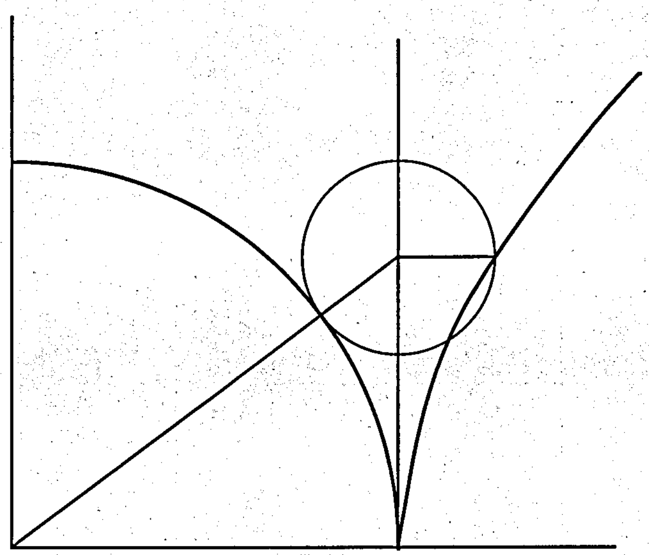


$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} f(x_i)\frac{b-a}{n} + \frac{1}{2}[f(x_n) - f(x_0)]\frac{b-a}{n}$$

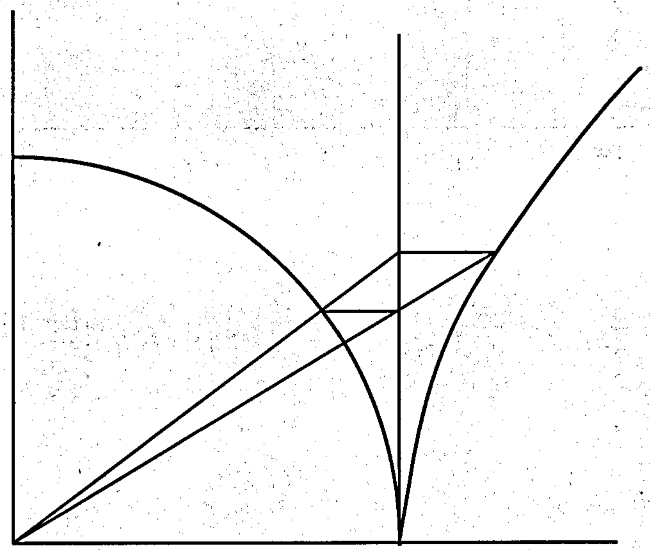
—Jesús Urías

Construction of a Hyperbola

I.

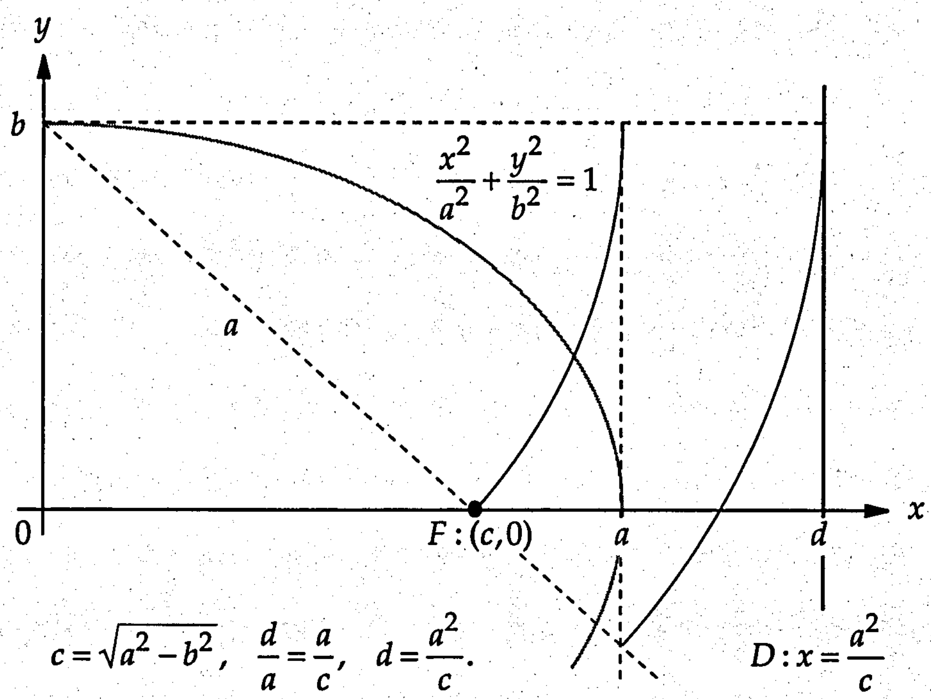


II.



—Ernest J. Eckert

The Focus and Directrix of an Ellipse

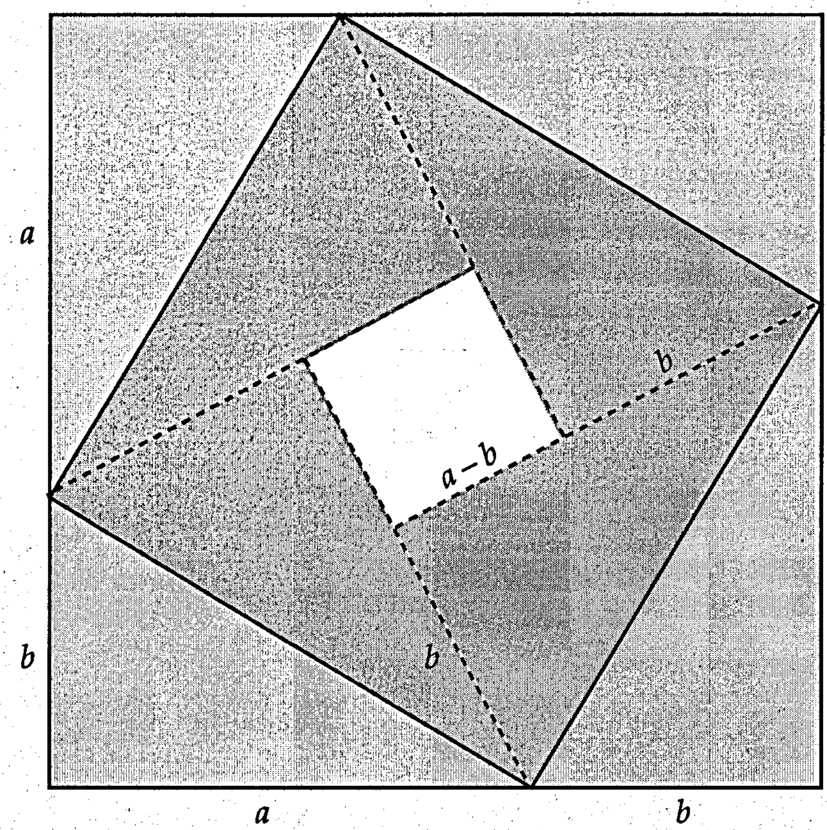


$c=\sqrt{a^2-b^2}, \quad \frac{d}{a}=\frac{a}{c}, \quad d=\frac{a^2}{c}.$

—Michel Bataille

Inequalities

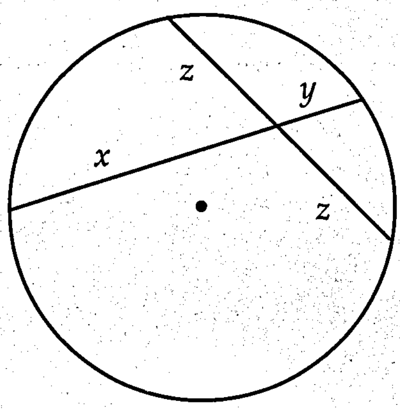
The Arithmetic Mean–Geometric Mean Inequality IV



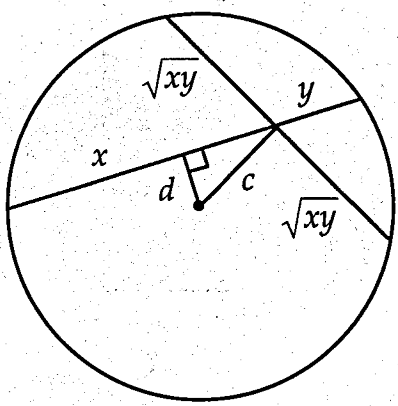
$$(a+b)^2 \geq 4ab \quad \Rightarrow \quad \frac{a+b}{2} \geq \sqrt{ab}$$

—Ayoub B. Ayoub

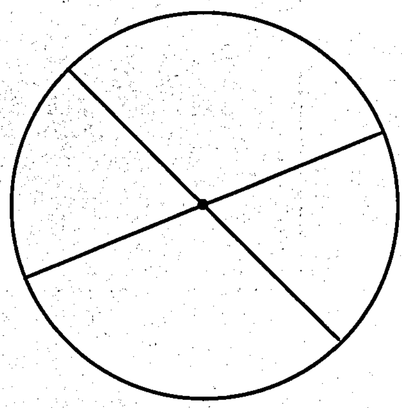
The Arithmetic Mean–Geometric Mean Inequality V



$$z^2 = xy$$



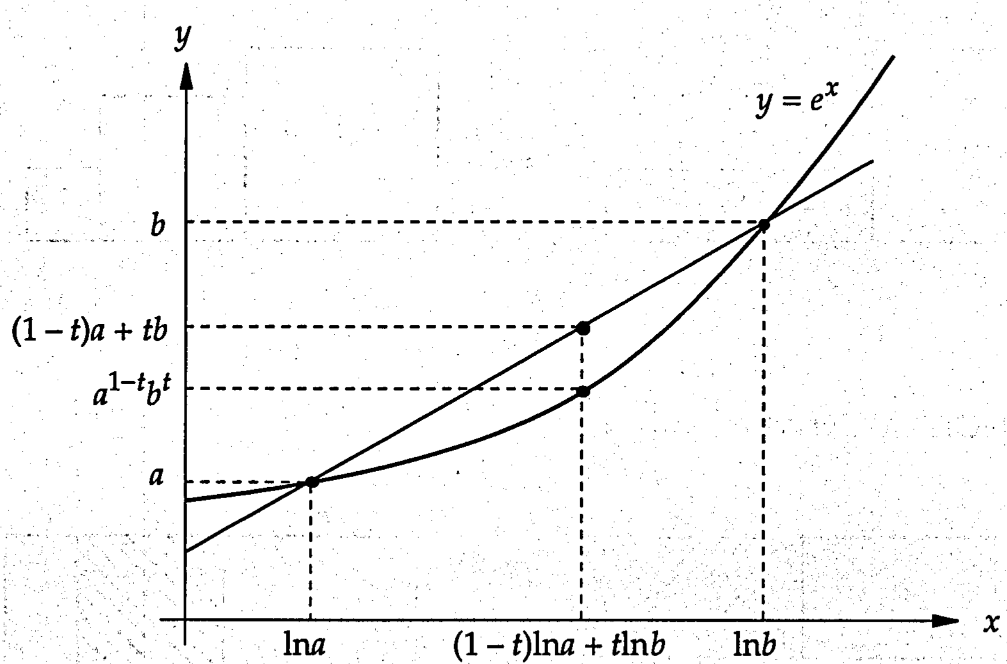
$$d < c \Rightarrow x + y > 2\sqrt{xy}$$



$$d = c = 0 \Rightarrow x + y = 2\sqrt{xy}$$

—Sidney H. Kung

The Arithmetic Mean–Geometric Mean Inequality VI



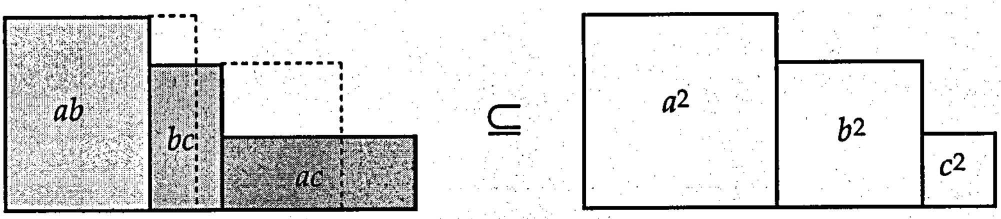
$$0 < a < b,\ 0 < t < 1 \Rightarrow (1-t)a + tb > a^{1-t}b^t$$

$$t = \frac{1}{2} \Rightarrow \frac{a+b}{2} > \sqrt{ab}$$

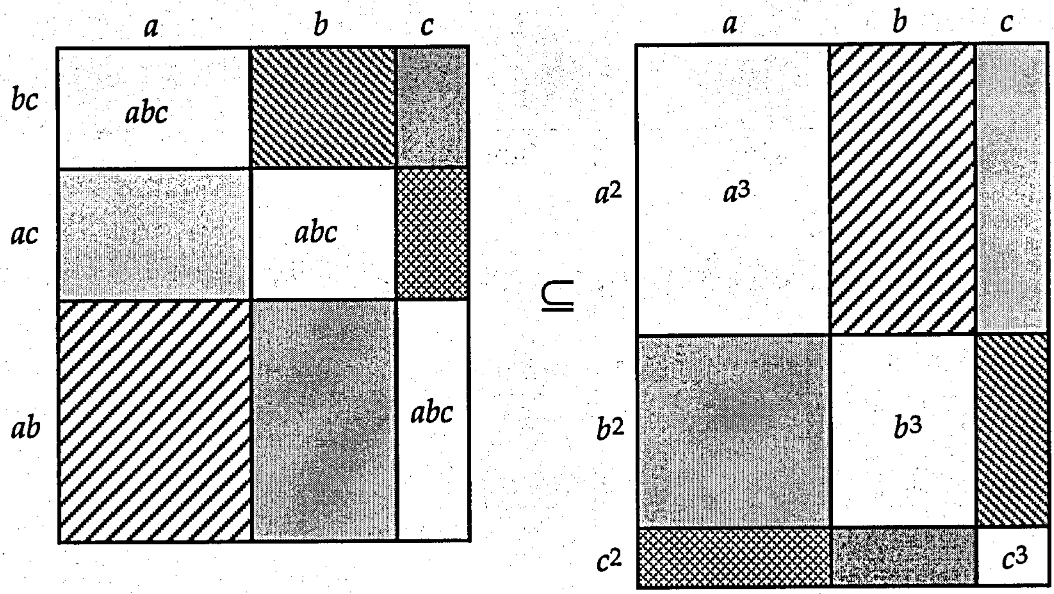
—Michael K. Brozinsky

The Arithmetic Mean–Geometric Mean Inequality for Three Positive Numbers

LEMMA: $ab + bc + ac \le a^2 + b^2 + c^2$



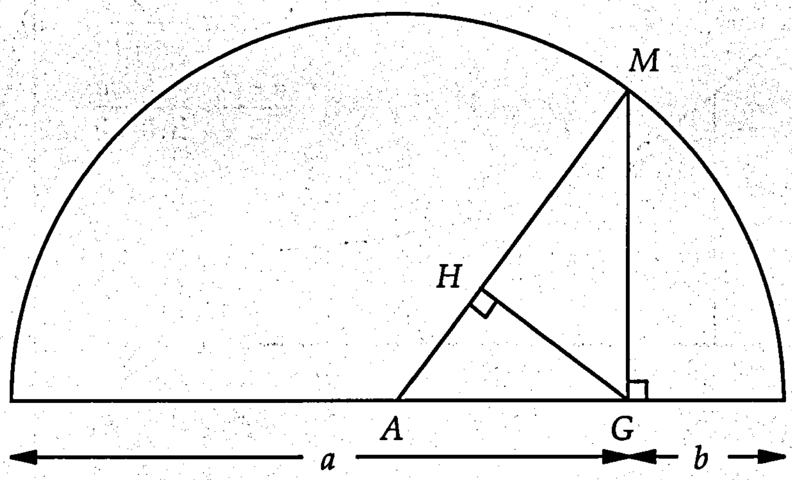
THEOREM: $3abc \le a^3 + b^3 + c^3$



—Claudi Alsina

The Arithmetic–Geometric–Harmonic Mean Inequality

$$a,b>0 \Rightarrow \frac{a+b}{2} \geq \sqrt{ab} \geq \frac{2ab}{a+b}$$

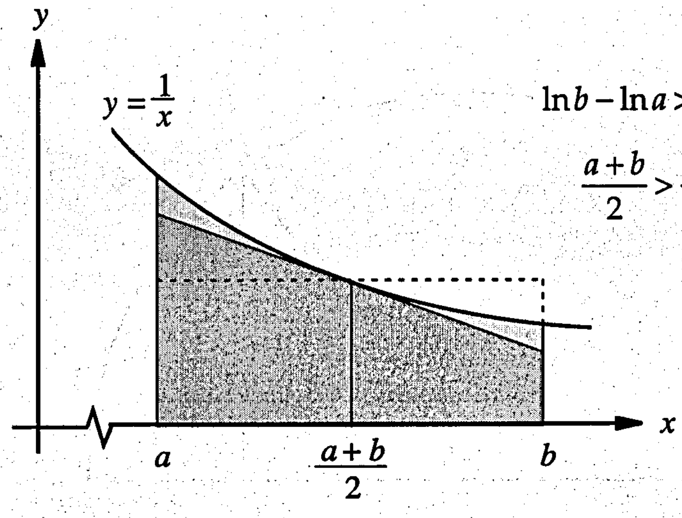


$$\overline{AM}=\frac{a+b}{2},\ \overline{GM}=\sqrt{ab},\ \overline{HM}=\frac{2ab}{a+b},$$
$$\overline{AM} \geq \overline{GM} \geq \overline{HM}.$$

—Pappus of Alexandria (circa A.D. 320)

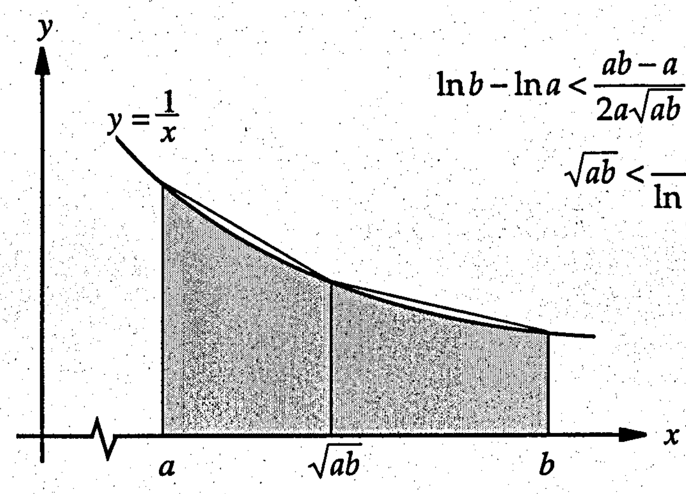
The Arithmetic–Logarithmic–Geometric Mean Inequality

$$b > a > 0 \Rightarrow \frac{a+b}{2} > \frac{b-a}{\ln b - \ln a} > \sqrt{ab}$$



$$\ln b - \ln a > \frac{2}{a+b}(b-a)$$

$$\frac{a+b}{2} > \frac{b-a}{\ln b - \ln a}$$



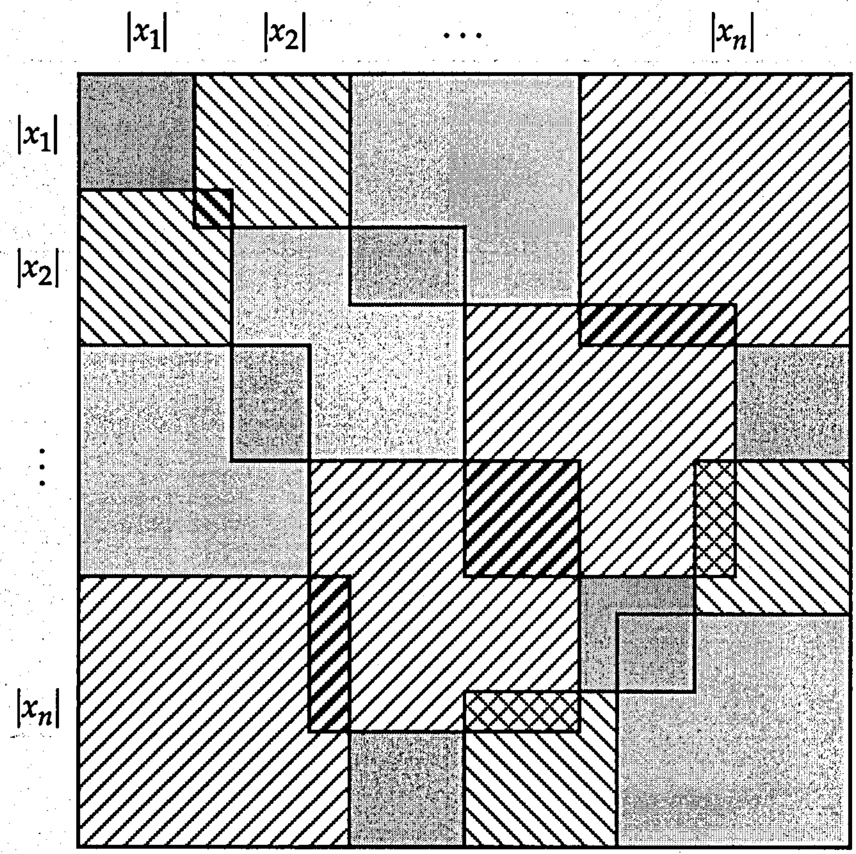
$$\ln b - \ln a < \frac{ab-a}{2a\sqrt{ab}} + \frac{a-ab}{2b\sqrt{ab}} = \frac{b-a}{\sqrt{ab}}$$

$$\sqrt{ab} < \frac{b-a}{\ln b - \ln a}$$

—RBN

The Mean of the Squares Exceeds the Square of the Mean

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 \geq \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2$$



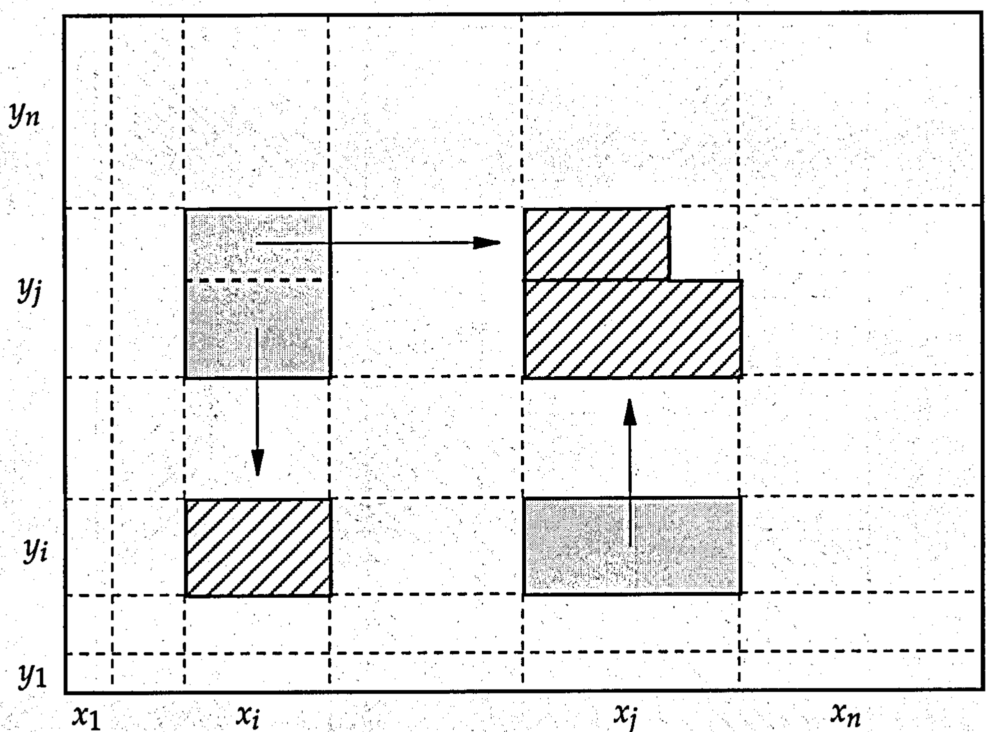
$$n\left(x_1^2 + x_2^2 + \cdots + x_n^2\right) \geq \left(|x_1| + |x_2| + \cdots + |x_n|\right)^2 \geq (x_1 + x_2 + \cdots + x_n)^2$$

$$\therefore \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n} \geq \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^2$$

—RBN

The Chebyshev Inequality for Positive Monotone Sequences

$$\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i \le n \sum_{i=1}^{n} x_i y_i$$



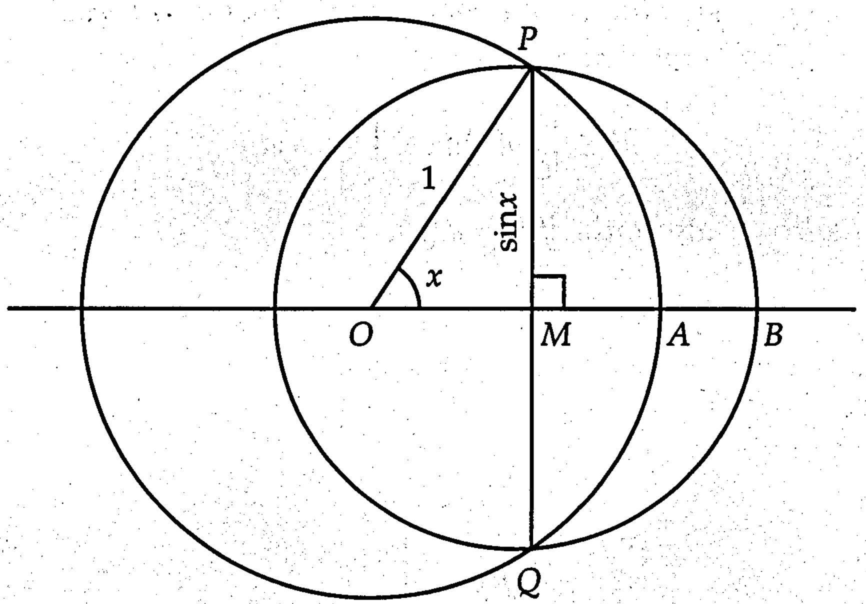
$$x_i < x_j \ \ \& \ \ y_i < y_j \Rightarrow x_i y_j + x_j y_i \le x_i y_i + x_j y_j$$

$$\therefore (x_1 + x_2 + \cdots + x_n)(y_1 + y_2 + \cdots + y_n) \le n(x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)$$

—RBN

Jordan's Inequality

$$0 \le x \le \frac{\pi}{2} \Rightarrow \frac{2x}{\pi} \le \sin x \le x$$



$$OB = OM + MP \ge OA \Rightarrow \overparen{PBQ} \ge \overparen{PAQ} \ge \overline{PQ}$$

$$\Rightarrow \pi \sin x \ge 2x \ge 2 \sin x$$

$$\Rightarrow \frac{2x}{\pi} \le \sin x \le x$$

—Feng Yuefeng

Young's Inequality

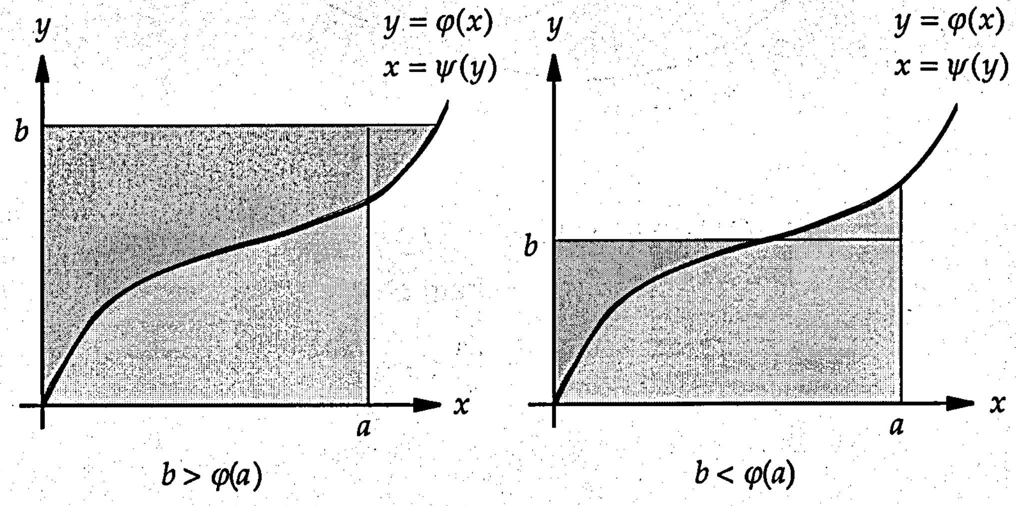
(W. H. Young, "On classes of summable functions and their Fourier series," *Proc. Royal Soc.* (A), 87 (1912) 225-229)

THEOREM: *Let φ and ψ be two functions, continuous, vanishing at the origin, strictly increasing, and inverse to each other. Then for $a,b \geq 0$ we have*

$$ab \leq \int_0^a \varphi(x)dx + \int_0^b \psi(y)dy$$

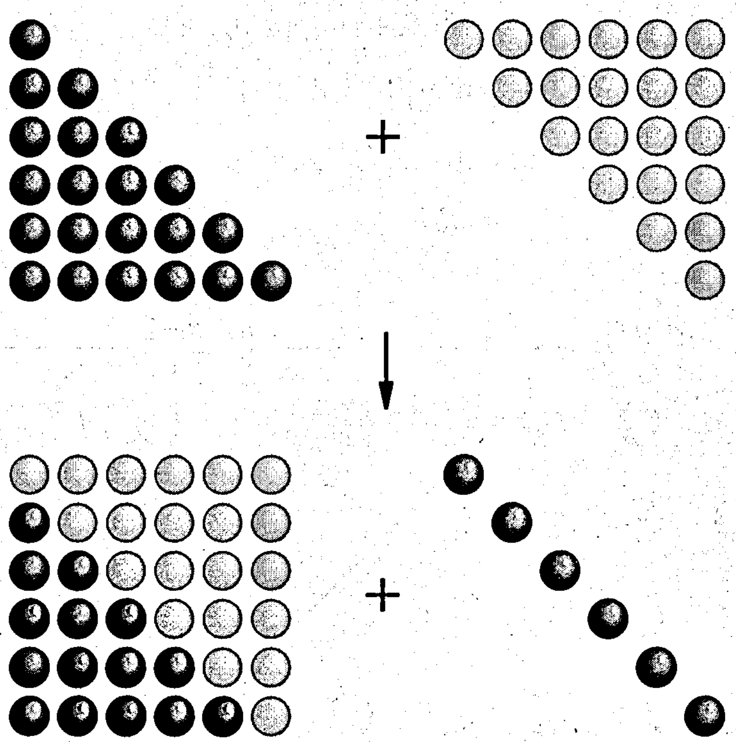
with equality if and only if $b = \varphi(a)$.

PROOF:



Integer Sums

Sums of Integers III



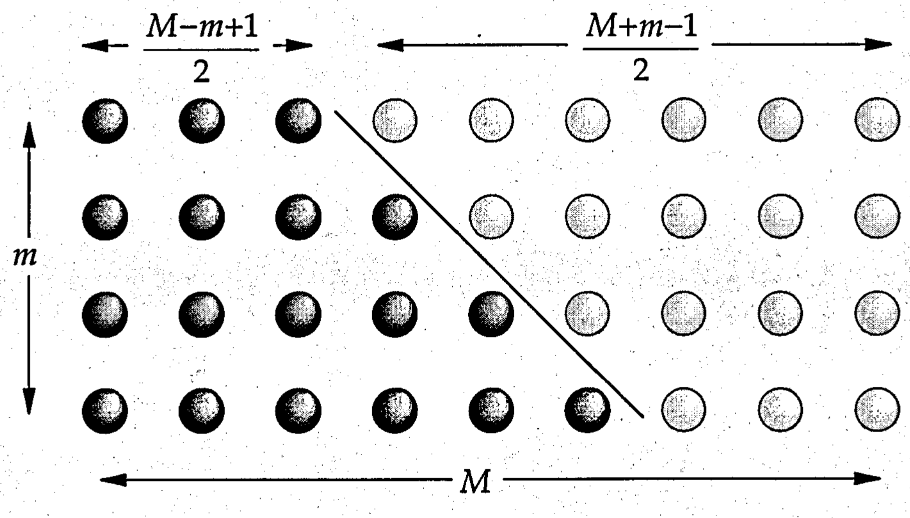
$$1+2+\cdots+n=\frac{1}{2}\left(n^2+n\right)$$

—S. J. Farlow

Sums of Consecutive Positive Integers

Every integer $N > 1$, not a power of two, can be expressed as the sum of two or more consecutive positive integers.

$$N = 2^n(2k+1) \quad (n \geq 0, k \geq 1),$$
$$m = \min\{2^{n+1}, 2k+1\},$$
$$M = \max\{2^{n+1}, 2k+1\},$$
$$2N = mM.$$



$$N = \left(\frac{M-m+1}{2}\right) + \left(\frac{M-m+1}{2}+1\right) + \cdots + \left(\frac{M+m-1}{2}\right).$$

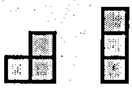
REFERENCES

1. P. Ross, Problem 1358, *Mathematics Magazine* 63 (1990), 350.
2. J. V. Wales, Jr., Solution to Problem 1358, *Mathematics Magazine* 64 (1991), 351.

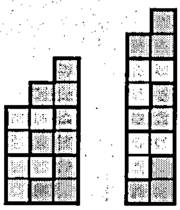
—C. L. Frenzen

Consecutive Sums of Consecutive Integers II

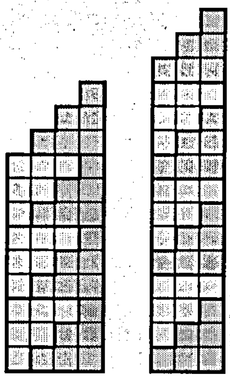
$T_k = 1 + 2 + \cdots + k \Rightarrow$



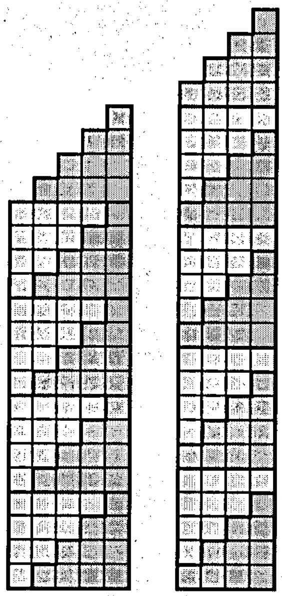
$1 + 2 = 3$
$= 3T_1$



$4 + 5 + 6 = 7 + 8$
$= 5T_2$



$9 + 10 + 11 + 12 = 13 + 14 + 15$
$= 7T_3$

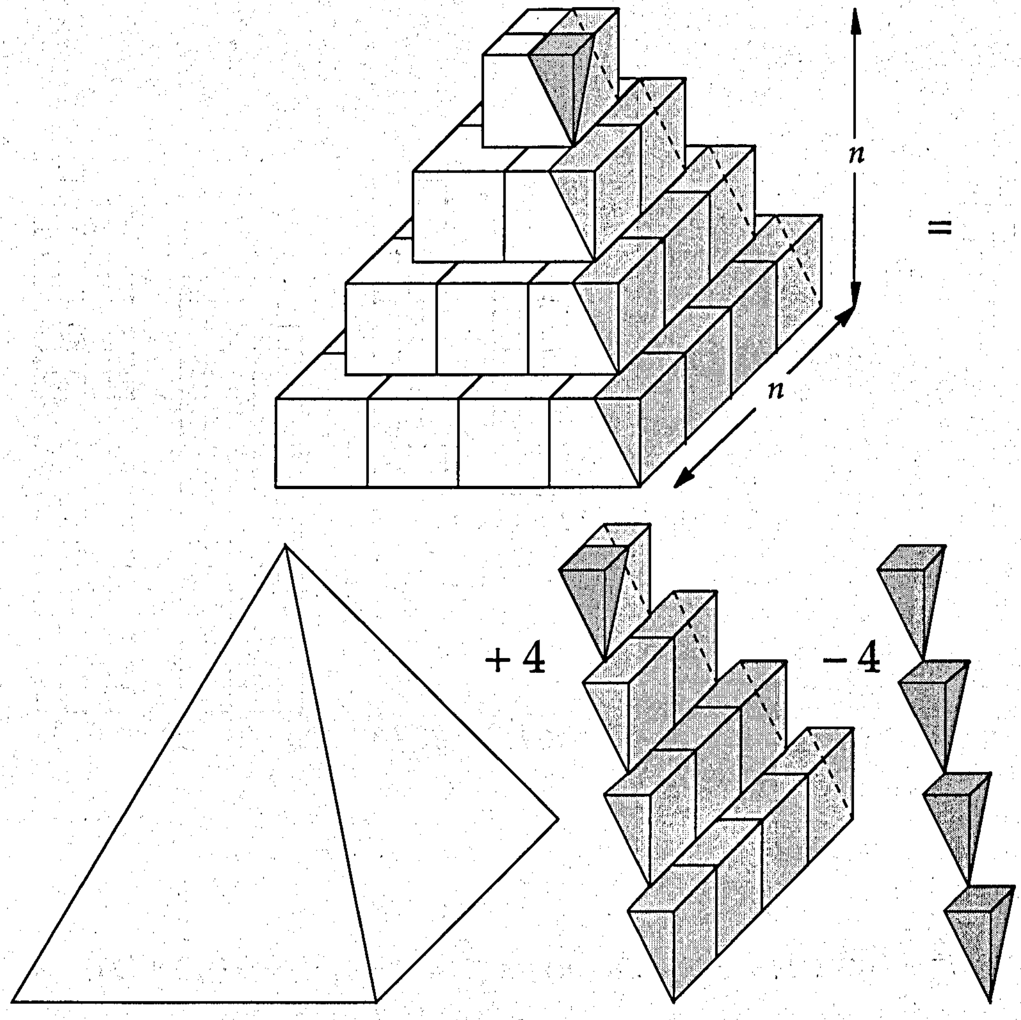


$16 + 17 + 18 + 19 + 20 = 21 + 22 + 23 + 24$
$= 9T_4$

...

$$n^2 + (n^2 + 1) + \cdots + (n^2 + n) = (n^2 + n + 1) + \cdots + (n^2 + 2n)$$
$$= (2n + 1)T_n$$

Sums of Squares VI



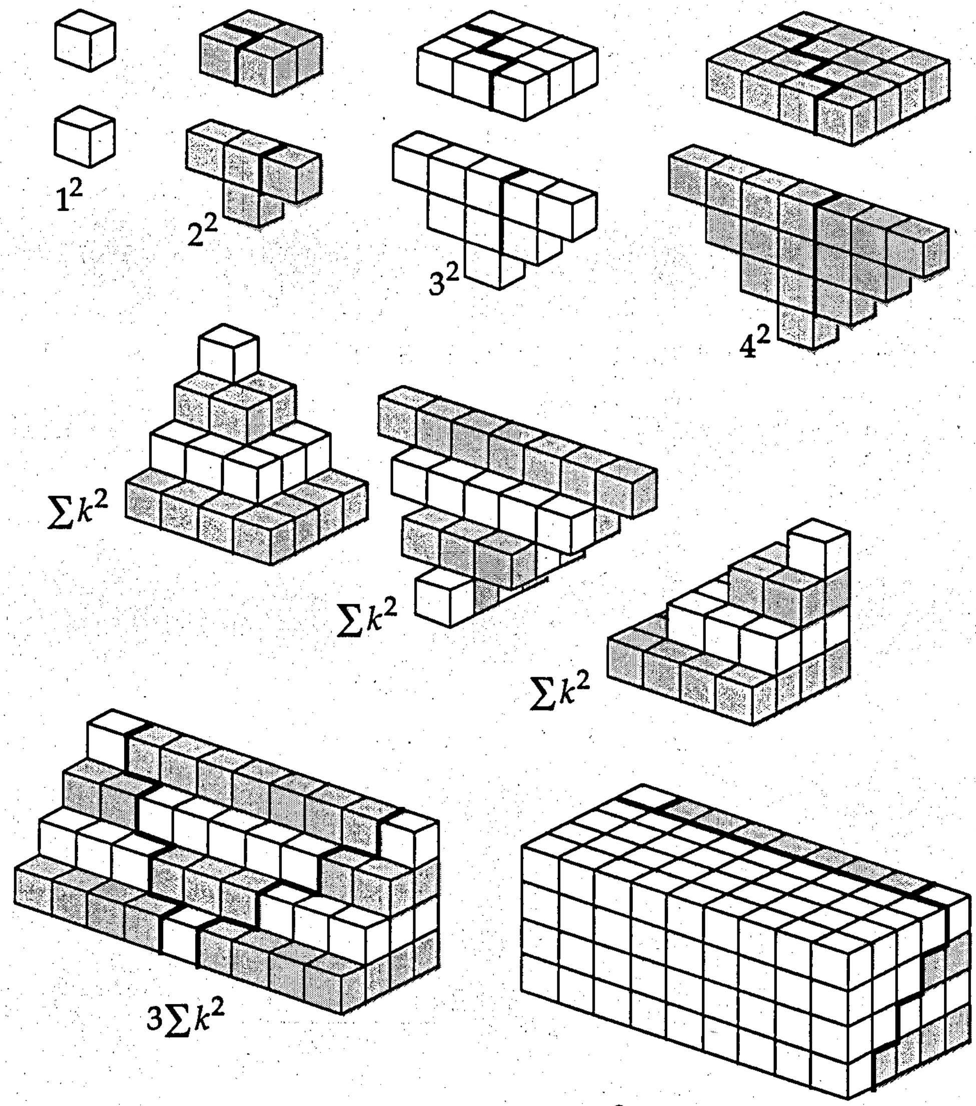
$$1^2+2^2+\cdots+n^2 = \frac{1}{3}n^2\cdot n + 4\cdot\frac{n(n+1)}{2}\cdot\frac{1}{4} - 4\cdot n\cdot\frac{1}{12}$$

$$= \frac{1}{6}n(n+1)(2n+1).$$

—I. A. Sakmar

Sums of Squares VII

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

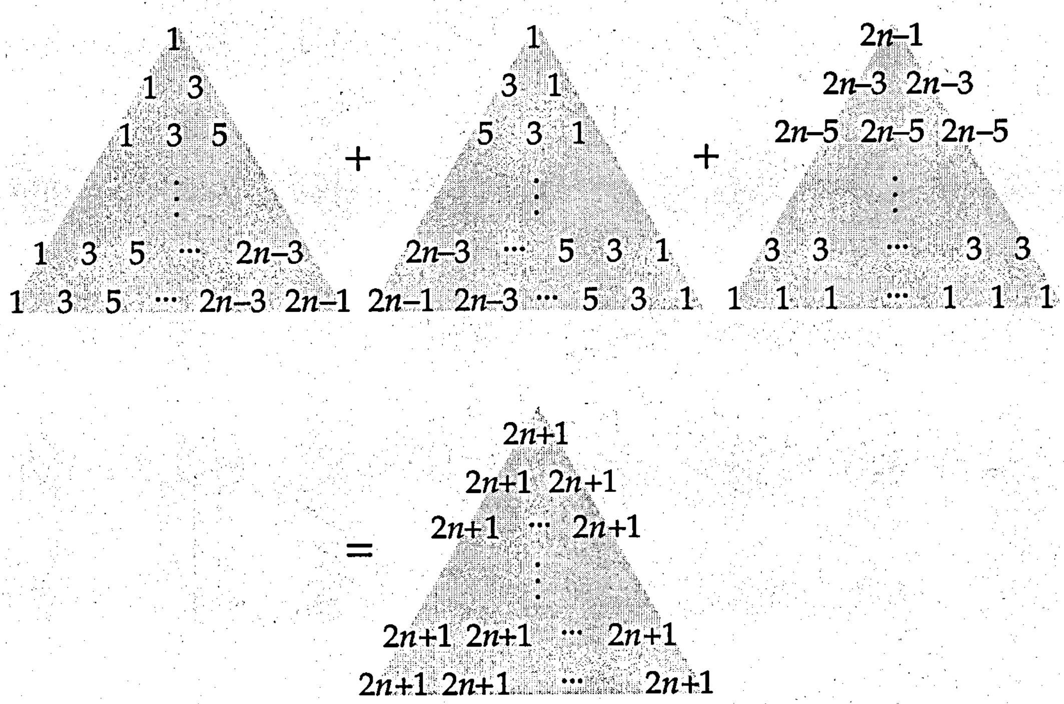


$$6\sum k^2 = n(n+1)(2n+1)$$

—Nanny Wermuth
and Hans-Jürgen Schuh

Sums of Squares VIII

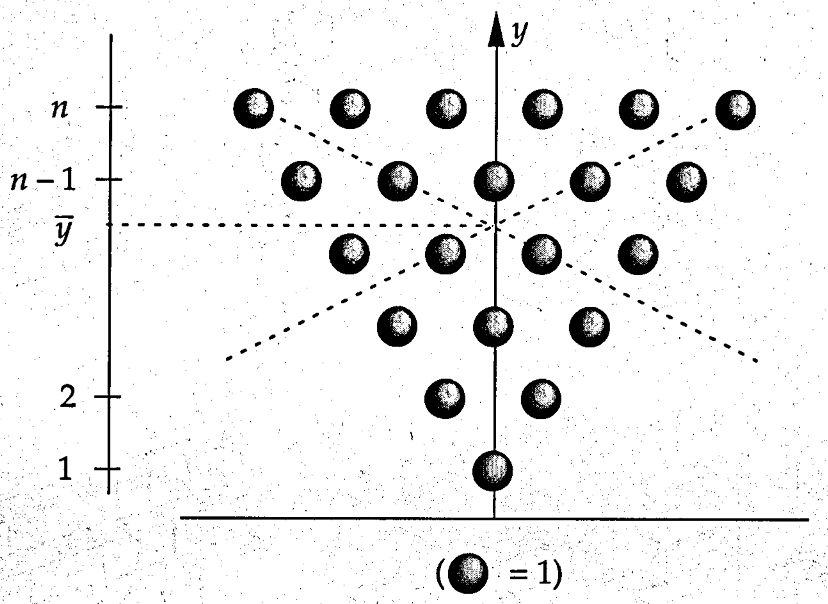
$$k^2 = 1 + 3 + \cdots + (2k-1) \Rightarrow \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$



$$3\left(1^2 + 2^2 + \cdots + n^2\right) = (2n+1)(1 + 2 + \cdots + n)$$

$$\therefore 1^2 + 2^2 + \cdots + n^2 = \frac{2n+1}{3} \cdot \frac{n(n+1)}{2}$$

Sums of Squares IX (via Centroids)



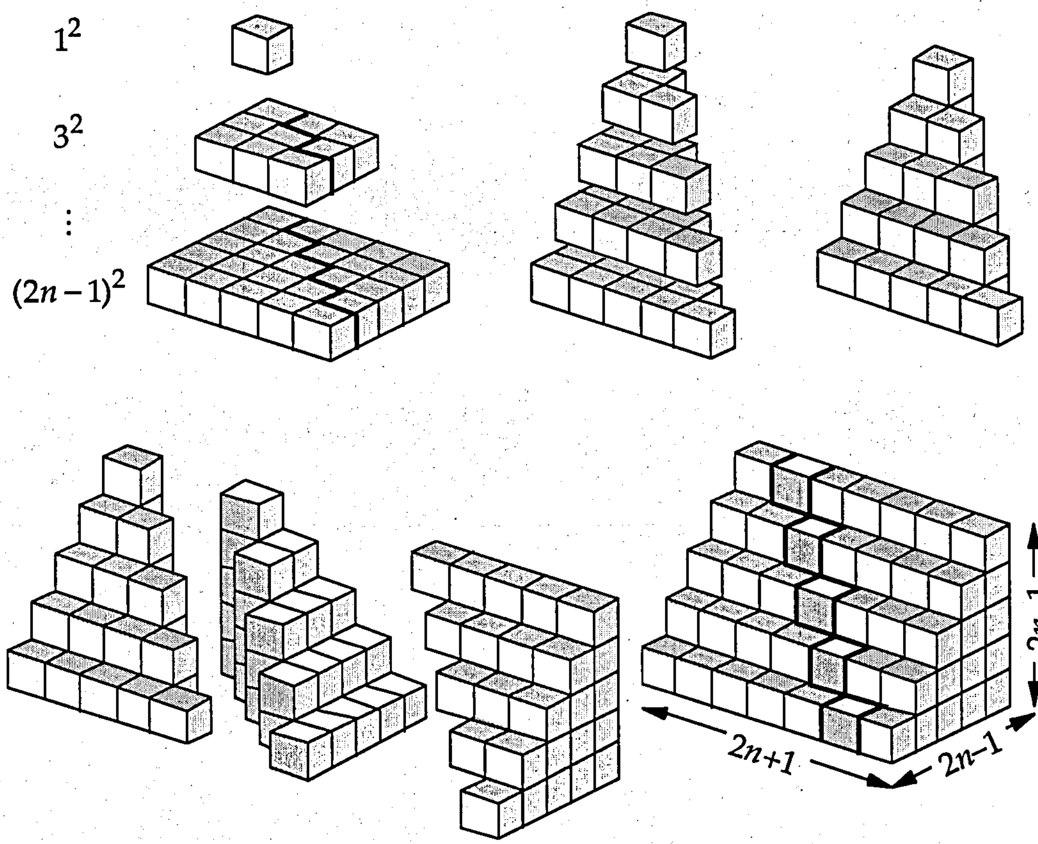
$$\bar{y} = 1 + \frac{2}{3}(n-1) = \frac{1\cdot 1 + 2\cdot 2 + \cdots + n\cdot n}{1 + 2 + \cdots + n}$$

$$\therefore 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)}{2}\cdot\frac{1}{3}(2n+1) = \frac{1}{6}n(n+1)(2n+1)$$

—Sidney H. Kung

Sums of Odd Squares

$$1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$



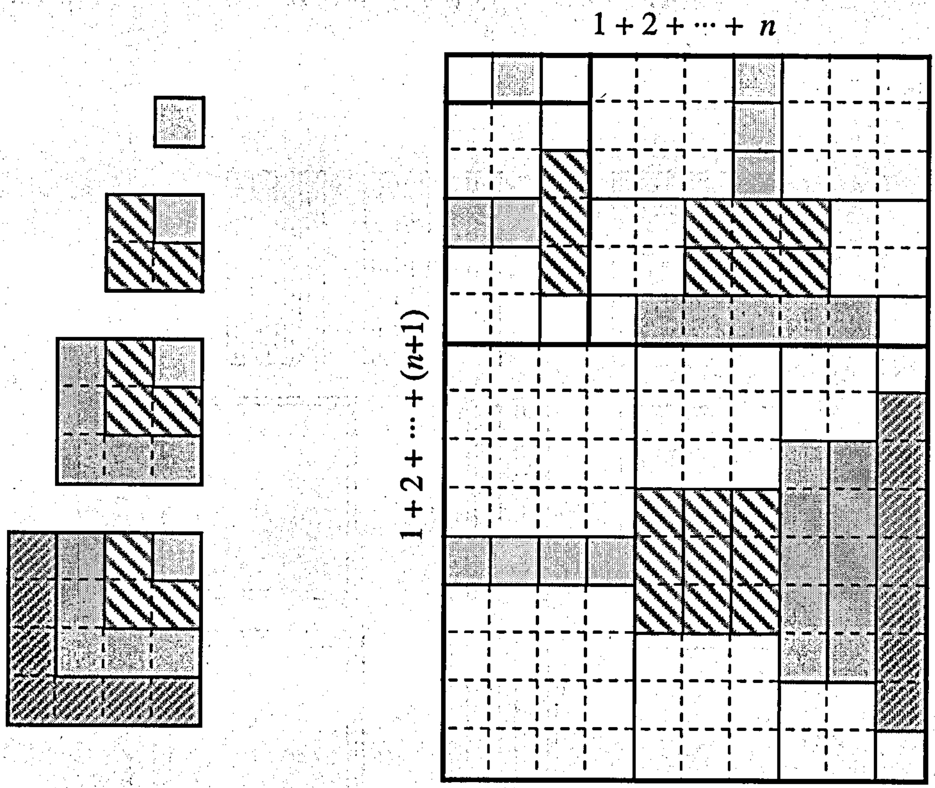
$$3 \times \left[1^2 + 3^2 + \cdots + (2n-1)^2\right] = [1 + 2 + \cdots + (2n-1)] \times (2n+1)$$

$$= \frac{(2n-1)(2n)(2n+1)}{2} = n(2n-1)(2n+1)$$

—RBN

Sums of Sums of Squares

$$\sum_{k=1}^{n}\sum_{i=1}^{k} i^2 = \frac{1}{3}\binom{n+1}{2}\binom{n+2}{2}$$



$$3\left(1^2\right)+3\left(1^2+2^2\right)+3\left(1^2+2^2+3^2\right)+\cdots+3\left(1^2+2^2+\cdots+n^2\right)=\binom{n+1}{2}\binom{n+2}{2}$$

—C. G. Wastun

Pythagorean Runs

$$3^2 + 4^2 = 5^2$$

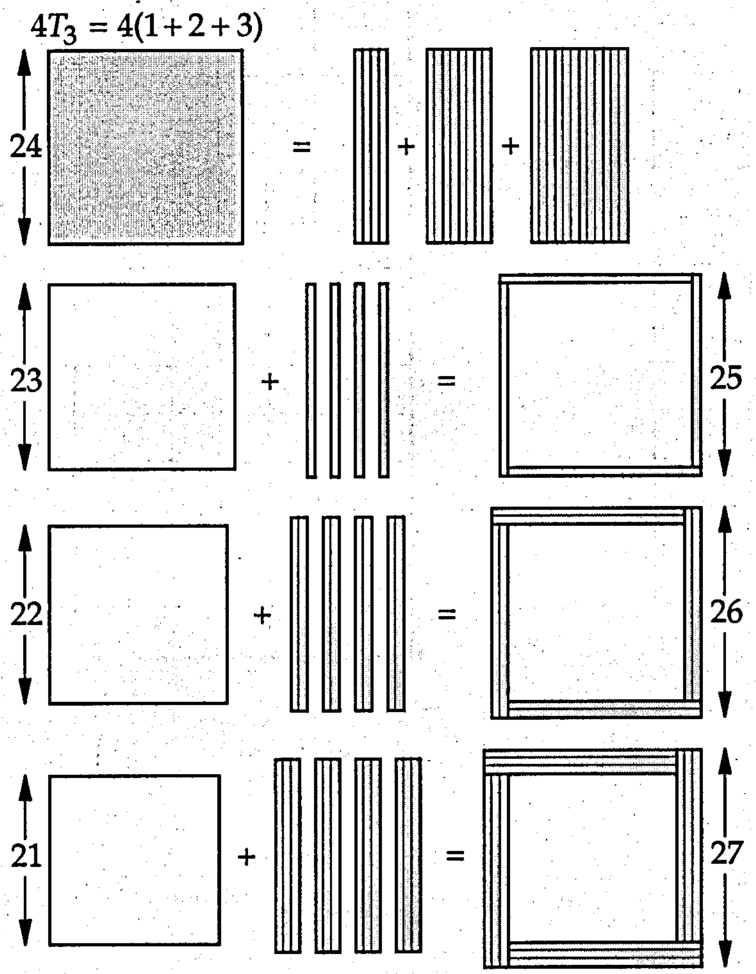
$$10^2 + 11^2 + 12^2 = 13^2 + 14^2$$

$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$$

$$\vdots$$

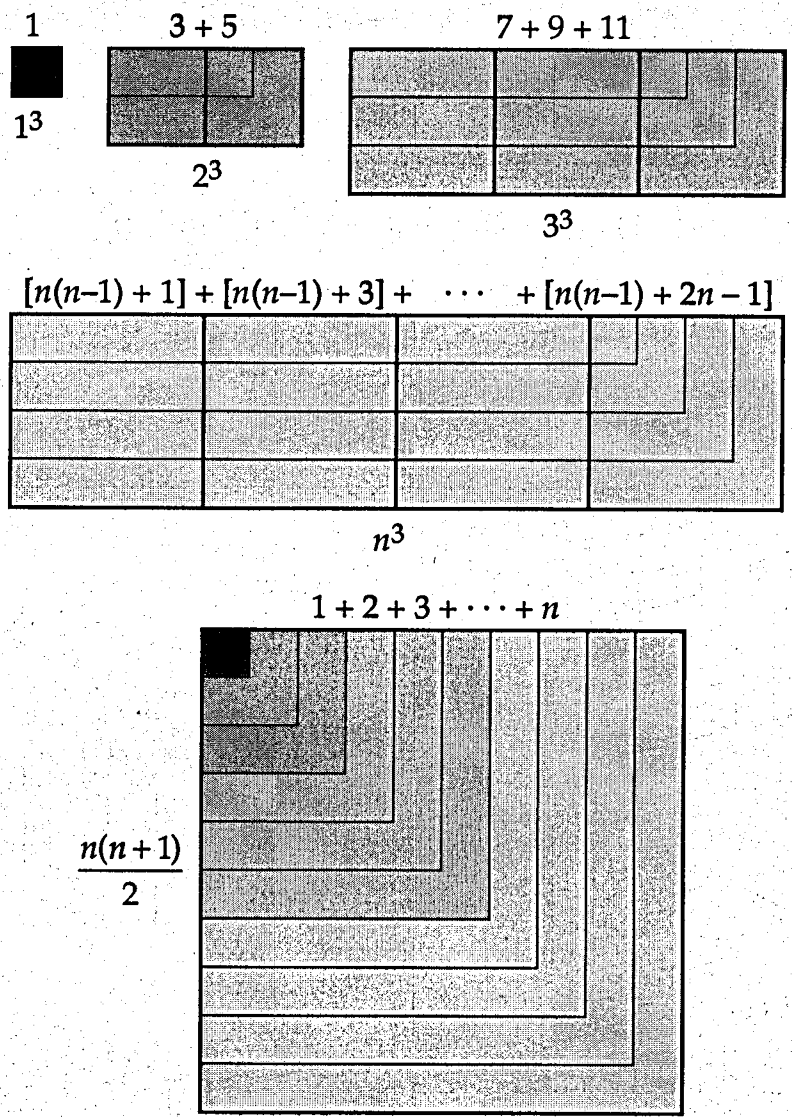
$$T_n = 1 + 2 + \cdots + n \Rightarrow (4T_n - n)^2 + \cdots + (4T_n)^2 = (4T_n + 1)^2 + \cdots + (4T_n + n)^2$$

e.g., $n = 3$:



—Michael Boardman

Sums of Cubes VII

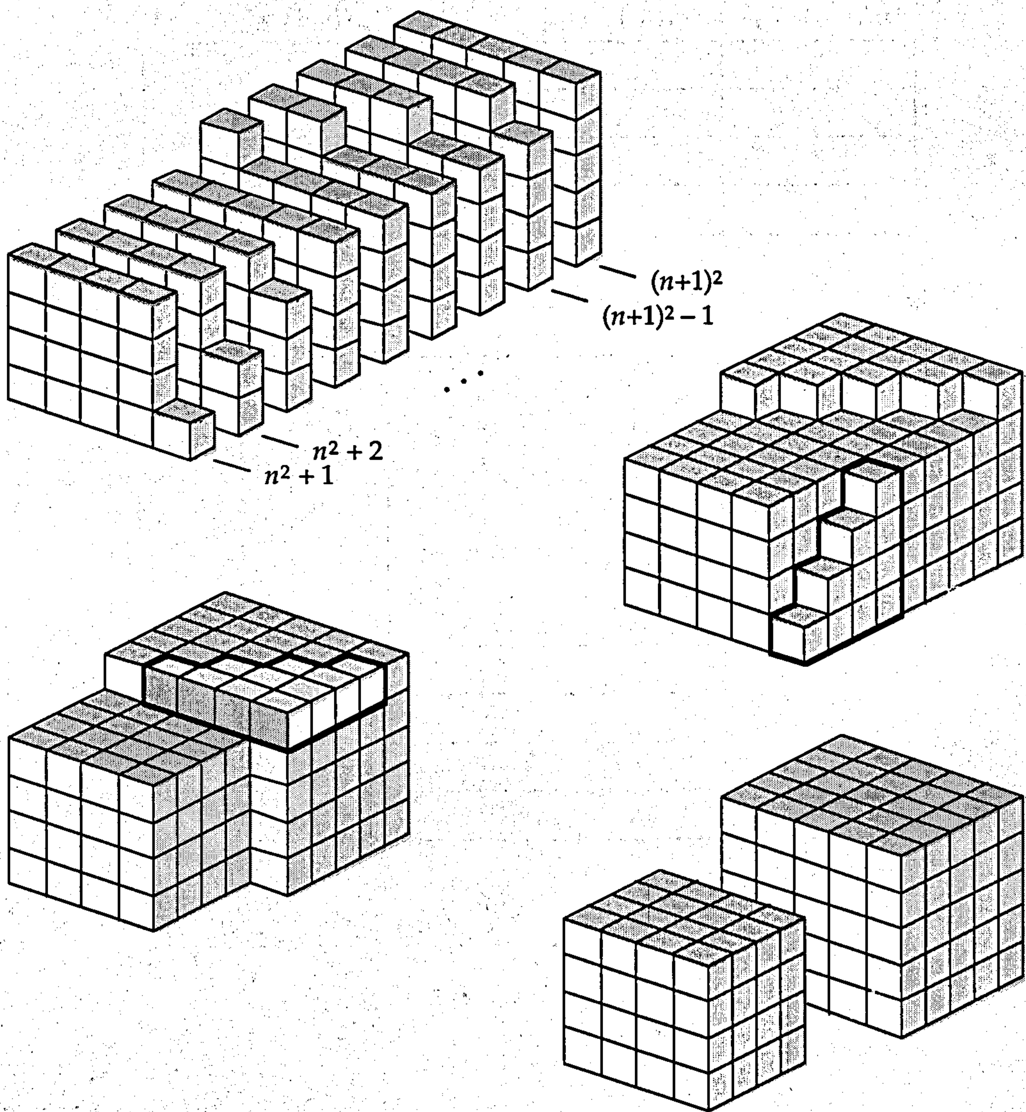


$$1^3 + 2^3 + \cdots + n^3 = 1 + 3 + 5 + \cdots + 2\frac{n(n+1)}{2} - 1 = \left[\frac{n(n+1)}{2}\right]^2$$

—Alfinio Flores

Sums of Integers as Sums of Cubes

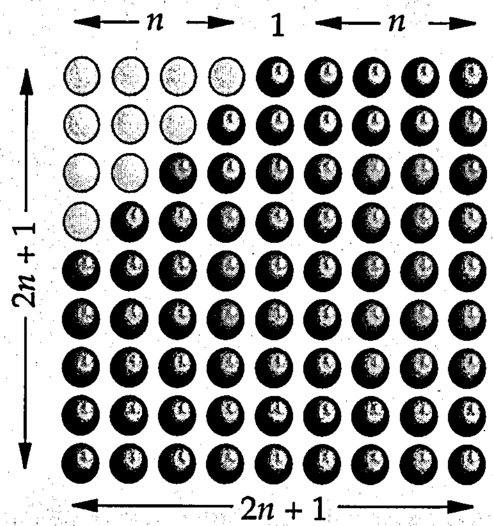
$$2+3+4=1+8$$
$$5+6+7+8+9=8+27$$
$$10+11+12+13+14+15+16=27+64$$
$$\vdots$$
$$(n^2+1)+(n^2+2)+\cdots+(n+1)^2=n^3+(n+1)^3$$

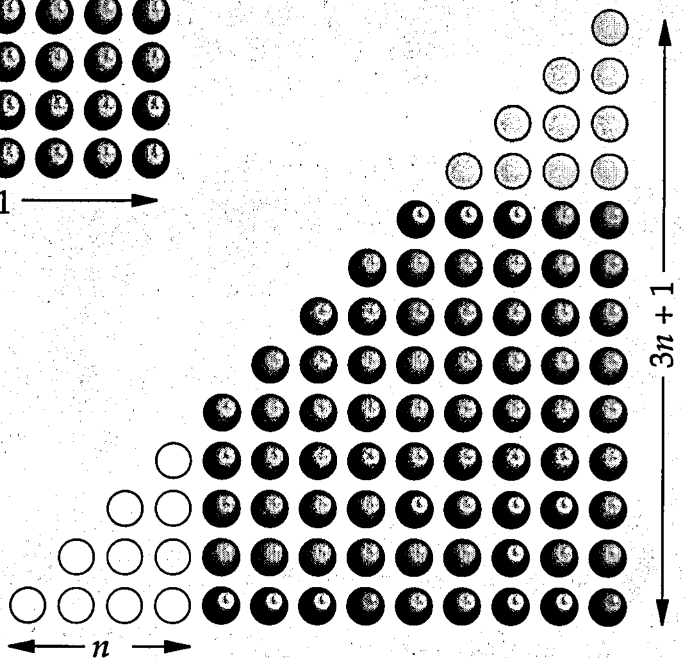


—RBN

The Square of Any Odd Number is the Difference Between Two Triangular Numbers

$$1+2+\cdots+k=T_k \Rightarrow (2n+1)^2 = T_{3n+1} - T_n$$

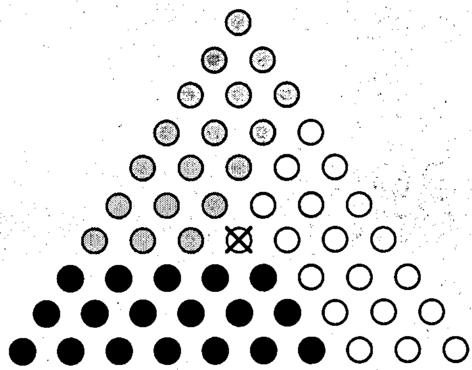




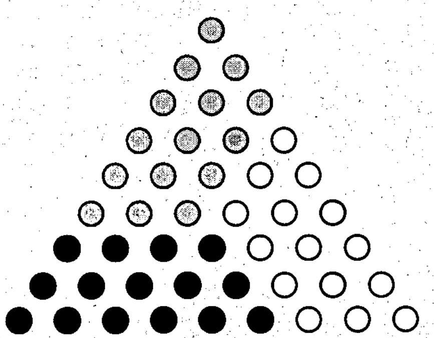
—RBN

Triangular Numbers Mod 3

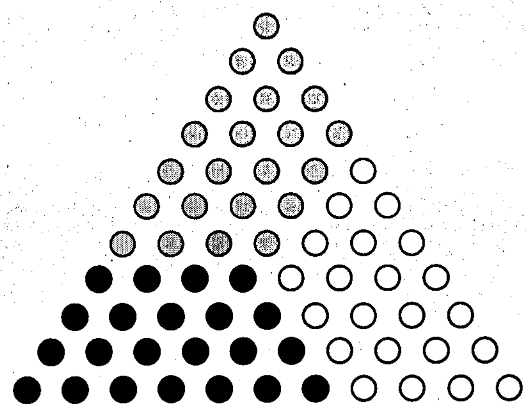
$$t_n = 1 + 2 + \cdots + n \Rightarrow \begin{cases} t_n \equiv 1 \bmod 3, & n \equiv 1 \bmod 3 \\ t_n \equiv 0 \bmod 3, & n \not\equiv 1 \bmod 3 \end{cases}$$



$$t_{3k+1} = 1 + 3(t_{2k+1} - t_{k+1})$$



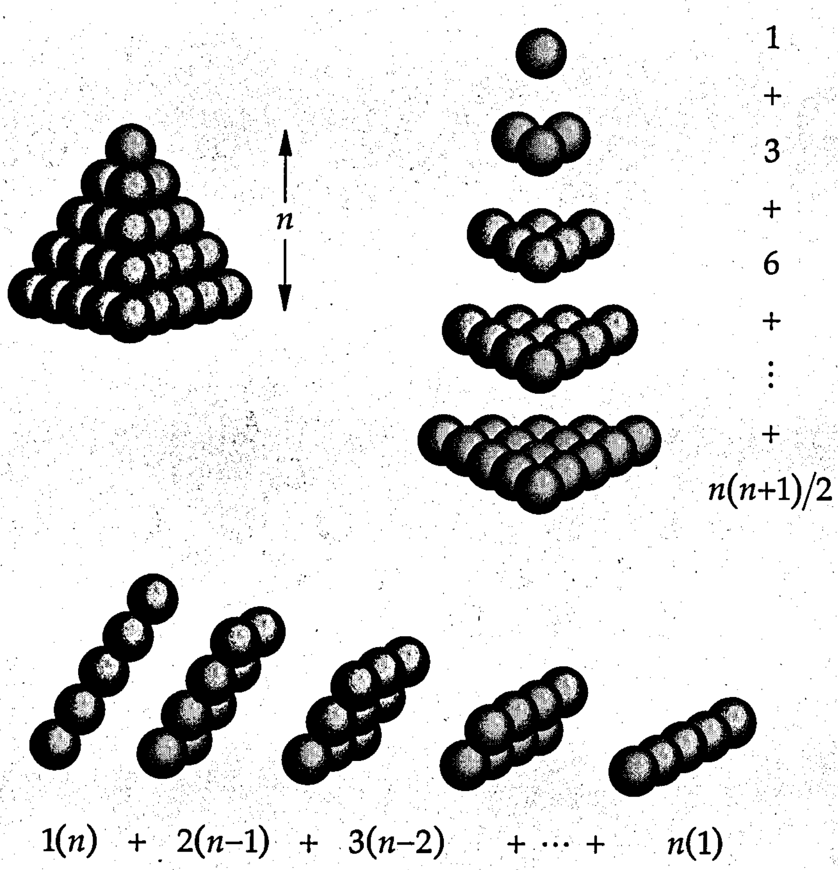
$$t_{3k} = 3(t_{2k} - t_k)$$



$$t_{3k+2} = 3(t_{2k+1} - t_k)$$

Sums of Triangular Numbers IV: Counting Cannonballs

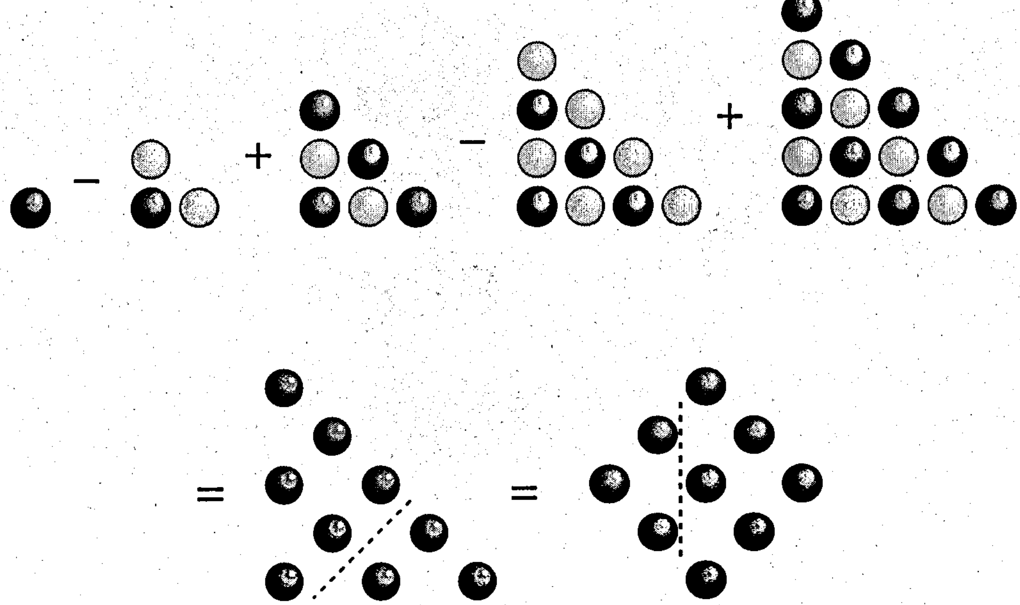
$$T_k = 1 + 2 + \cdots + k \Rightarrow \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} k(n-k+1)$$



—Deanna B. Haunsperger
and Stephen F. Kennedy

Alternating Sums of Triangular Numbers

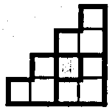
$$T_k = 1 + 2 + \cdots + k \Rightarrow \sum_{k=1}^{2n-1} (-1)^{k+1} T_k = n^2$$



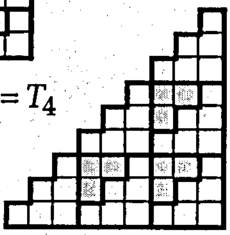
—RBN

The Sum of the Squares of Consecutive Triangular Numbers is Triangular

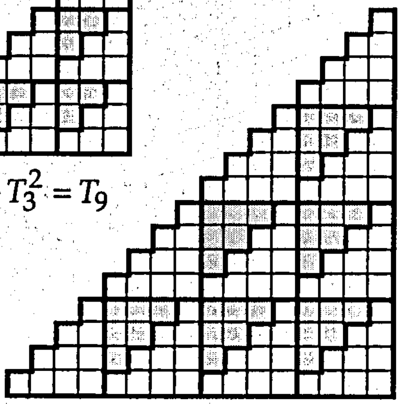
$$T_n = 1 + 2 + \cdots + n \Rightarrow T_{n-1}^2 + T_n^2 = T_{n^2}$$



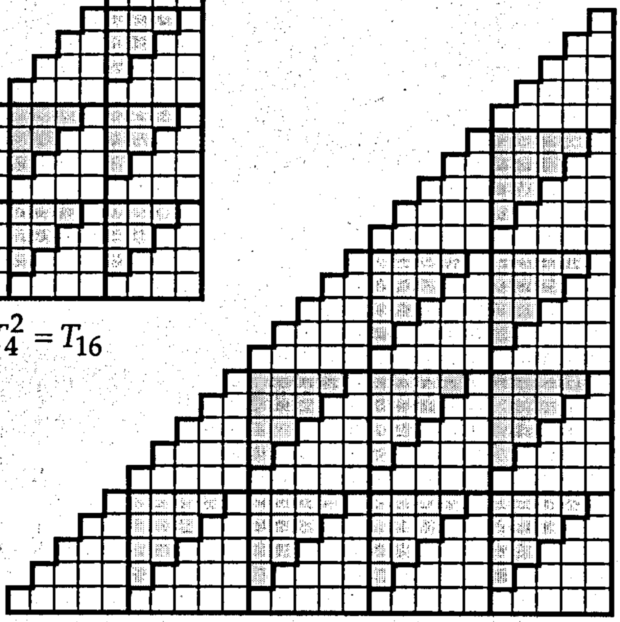
$T_1^2 + T_2^2 = T_4$



$T_2^2 + T_3^2 = T_9$

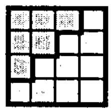


$T_3^2 + T_4^2 = T_{16}$



$T_4^2 + T_5^2 = T_{25}$

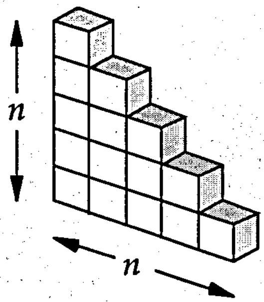
NOTE:
This is a companion result to the more familiar $T_{n-1} + T_n = n^2$:



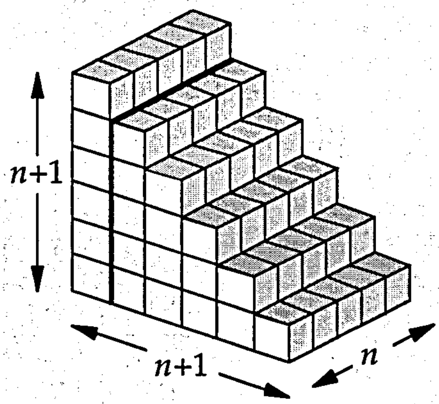
—RBN

Recursion for Triangular Numbers

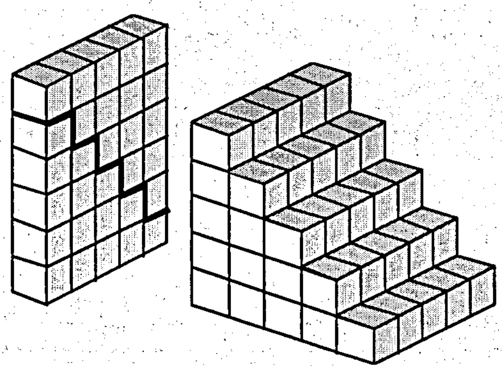
$$T_k = 1 + 2 + \cdots + k \Rightarrow T_{n+1} = \frac{n+2}{n} T_n$$



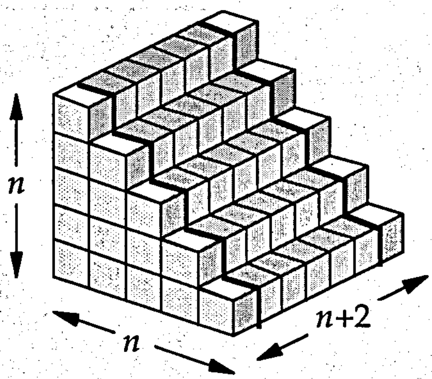
T_n



$n \cdot T_{n+1}$



$2 \cdot T_n + n \cdot T_n$



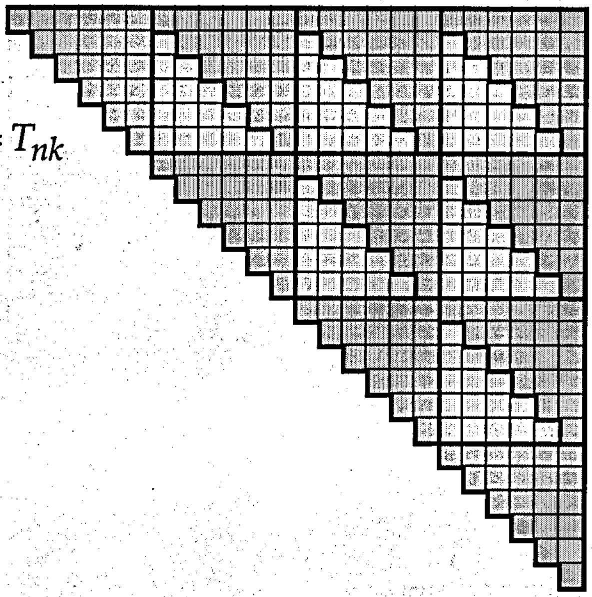
$(n+2) \cdot T_n$

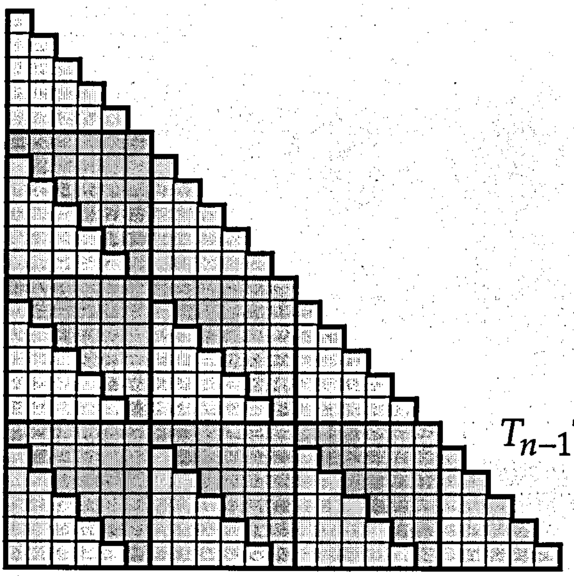
$$n \cdot T_{n+1} = (n+2) \cdot T_n \Rightarrow T_{n+1} = \frac{n+2}{n} T_n$$

Identities for Triangular Numbers II

$$T_n = 1 + 2 + \cdots + n \Rightarrow$$

$$T_n T_k + T_{n-1} T_{k-1} = T_{nk}$$





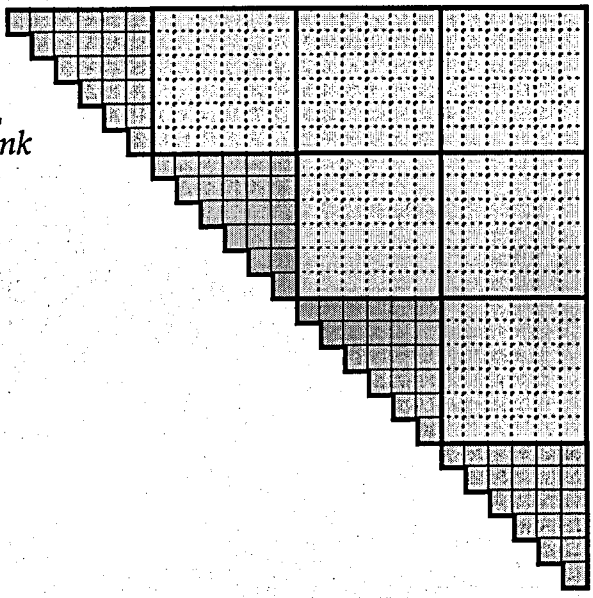
$$T_{n-1} T_k + T_n T_{k-1} = T_{nk-1}$$

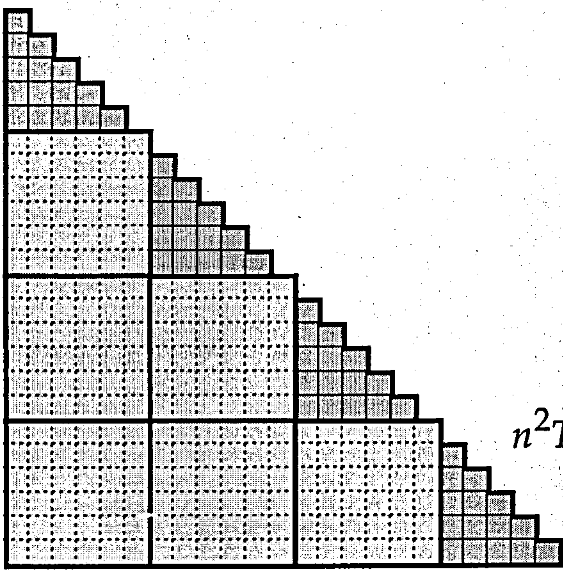
—RBN

Identities for Triangular Numbers III

$$T_n = 1 + 2 + \cdots + n \Rightarrow$$

$$n^2 T_{k-1} + kT_n = T_{nk}$$



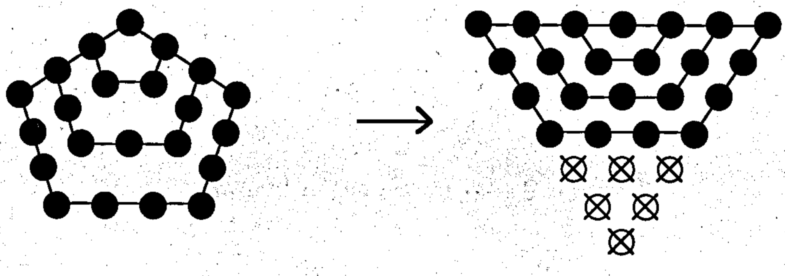


$$n^2 T_{k-1} + kT_{n-1} = T_{nk-1}$$

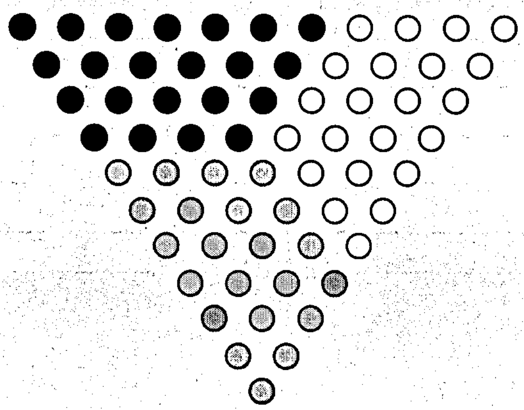
—James O. Chilaka

Identities for Pentagonal Numbers

$$\left.\begin{aligned} P_n &= 1+4+7+\cdots+(3n-2) \\ T_n &= 1+2+3+\cdots+n \end{aligned}\right\} \Rightarrow$$

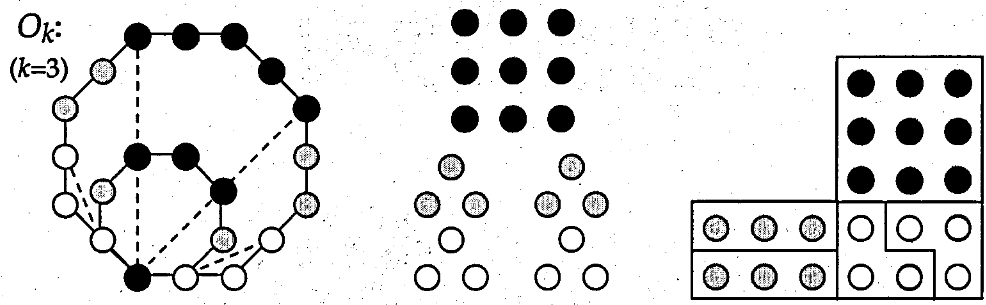


$$P_n = T_{2n-1} - T_{n-1}$$

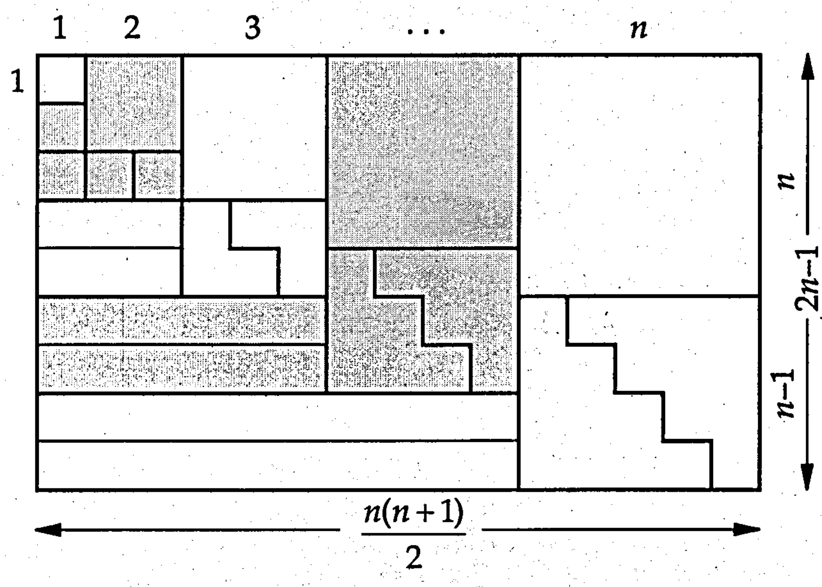


$$P_n = \frac{1}{3}T_{3n-1}$$

Sums of Octagonal Numbers



$$T_k = 1+2+\cdots+k \Rightarrow O_k = k^2 + 4T_{k-1}$$

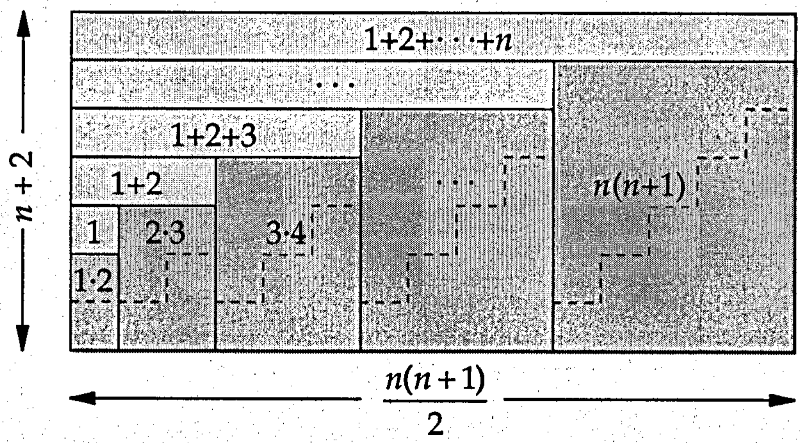


$$\sum_{k=1}^{n} O_k = 1+8+21+40+\cdots+(n^2+4T_{n-1}) = \frac{n(n+1)(2n-1)}{2}$$

—James O. Chilaka

Sums of Products of Consecutive Integers I

$$\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$$



$T_k = 1 + 2 + \cdots + k \quad \Rightarrow$

$$1\cdot 2 + 2\cdot 3 + \cdots + n(n+1) + \left(T_1 + T_2 + \cdots + T_n\right) = \frac{n(n+1)(n+2)}{2},$$

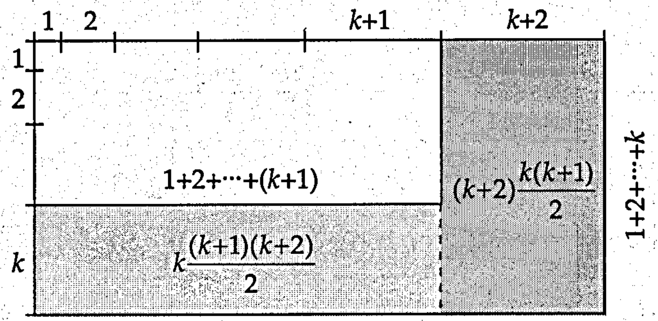
$$T_1 + T_2 + \cdots + T_n = \frac{1}{2}(1\cdot 2 + 2\cdot 3 + \cdots + n(n+1)),$$

$$\therefore \frac{3}{2}(1\cdot 2 + 2\cdot 3 + \cdots + n(n+1)) = \frac{n(n+1)(n+2)}{2}.$$

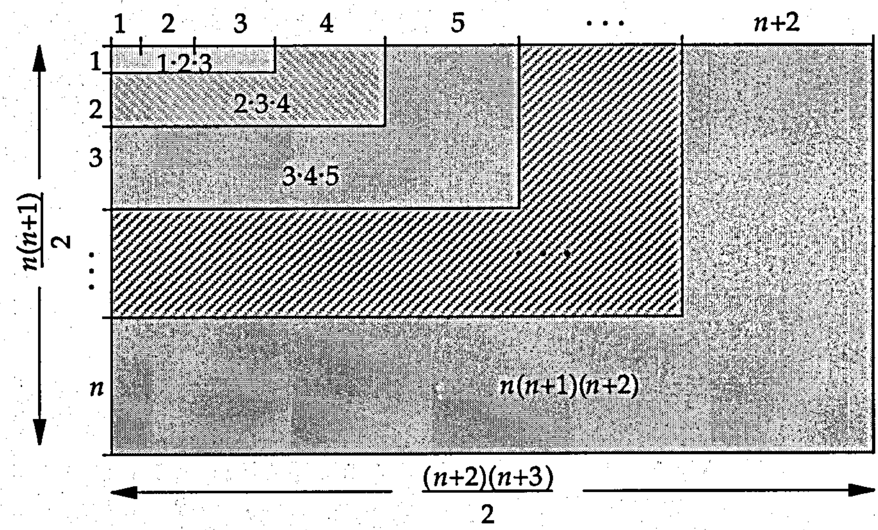
—James O. Chilaka

Sums of Products of Consecutive Integers II

$$\sum_{k=1}^{n} k(k+1)(k+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$



$$k\frac{(k+1)(k+2)}{2} + (k+2)\frac{k(k+1)}{2} = k(k+1)(k+2)$$

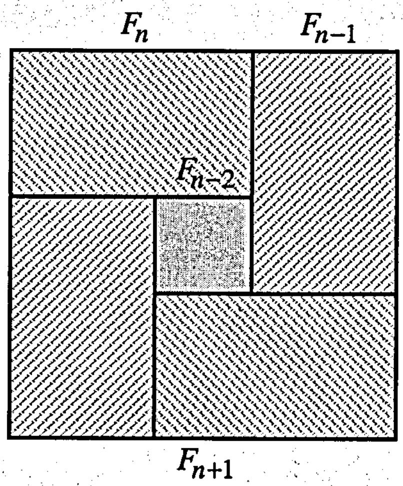


$$1\cdot2\cdot3 + 2\cdot3\cdot4 + \cdots + n(n+1)(n+2)$$
$$= \frac{n(n+1)}{2} \times \frac{(n+2)(n+3)}{2} = \frac{n(n+1)(n+2)(n+3)}{4}$$

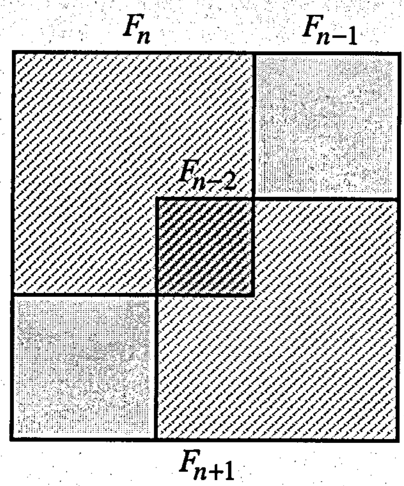
—James O. Chilaka

Fibonacci Identities

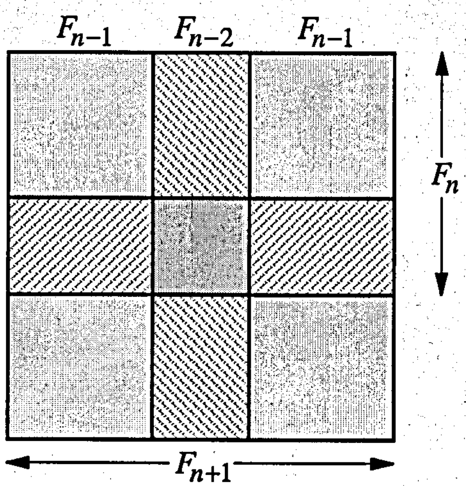
$F_1 = F_2 = 1, \quad F_n = F_{n-1} + F_{n-2} \Rightarrow$



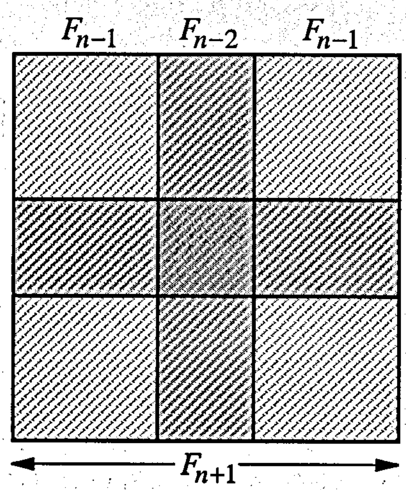
$$F_{n+1}^2 = 4F_nF_{n-1} + F_{n-2}^2$$



$$F_{n+1}^2 = 2F_n^2 + 2F_{n-1}^2 - F_{n-2}^2$$



$$F_{n+1}^2 = 4F_{n-1}^2 + 4F_{n-1}F_{n-2} + F_{n-2}^2$$

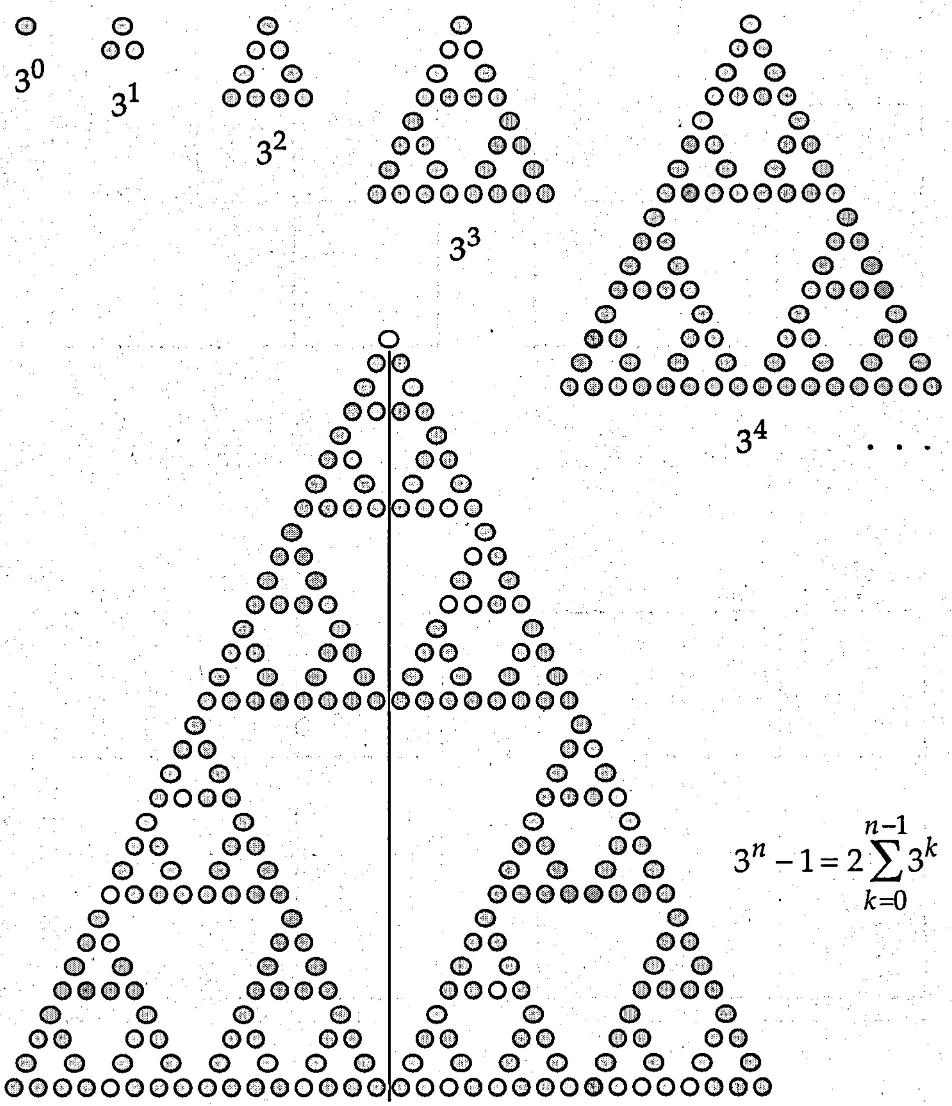


$$F_{n+1}^2 = 4F_n^2 - 4F_{n-1}F_{n-2} - 3F_{n-2}^2$$

—Alfred Brousseau

Sums of Powers of Three

$$\sum_{k=0}^{n-1} 3^k = \frac{3^n - 1}{2}$$

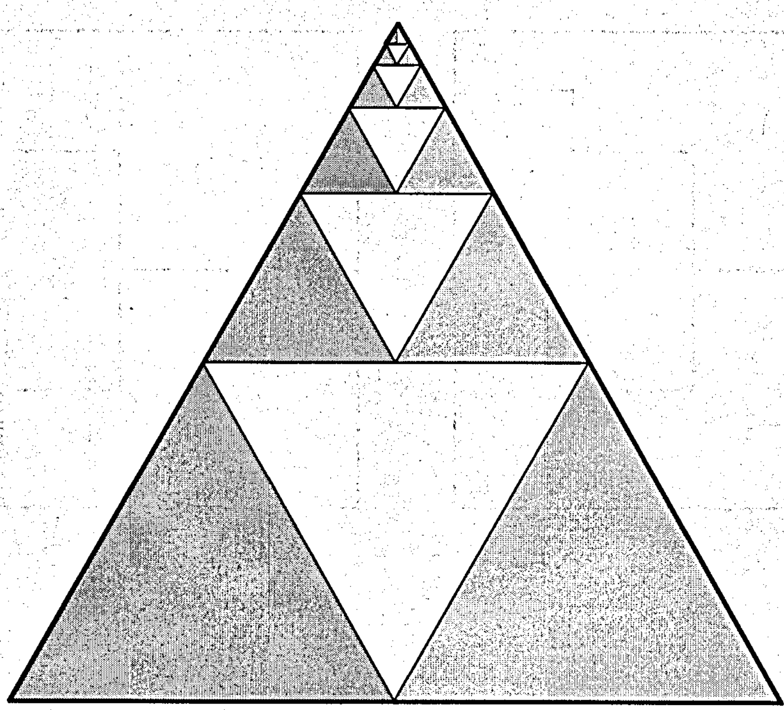


—David B. Sher

Infinite Series, Linear Algebra, & Other Topics

A Geometric Series

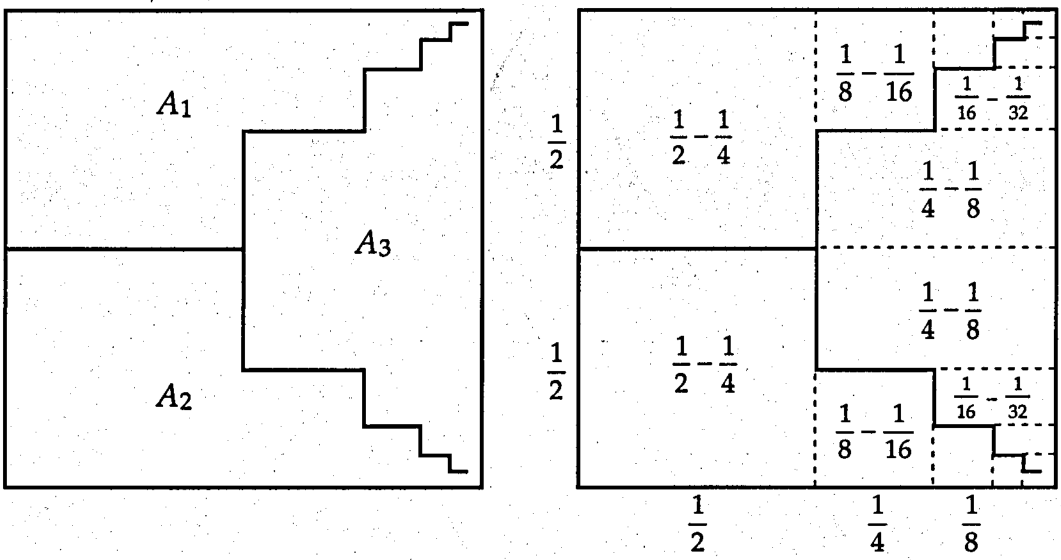
$$\frac{1}{4}+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^3+\cdots=\frac{1}{3}$$



—Rick Mabry

An Alternating Series

$$\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\frac{1}{32}-\frac{1}{64}+\cdots=\frac{1}{3}$$



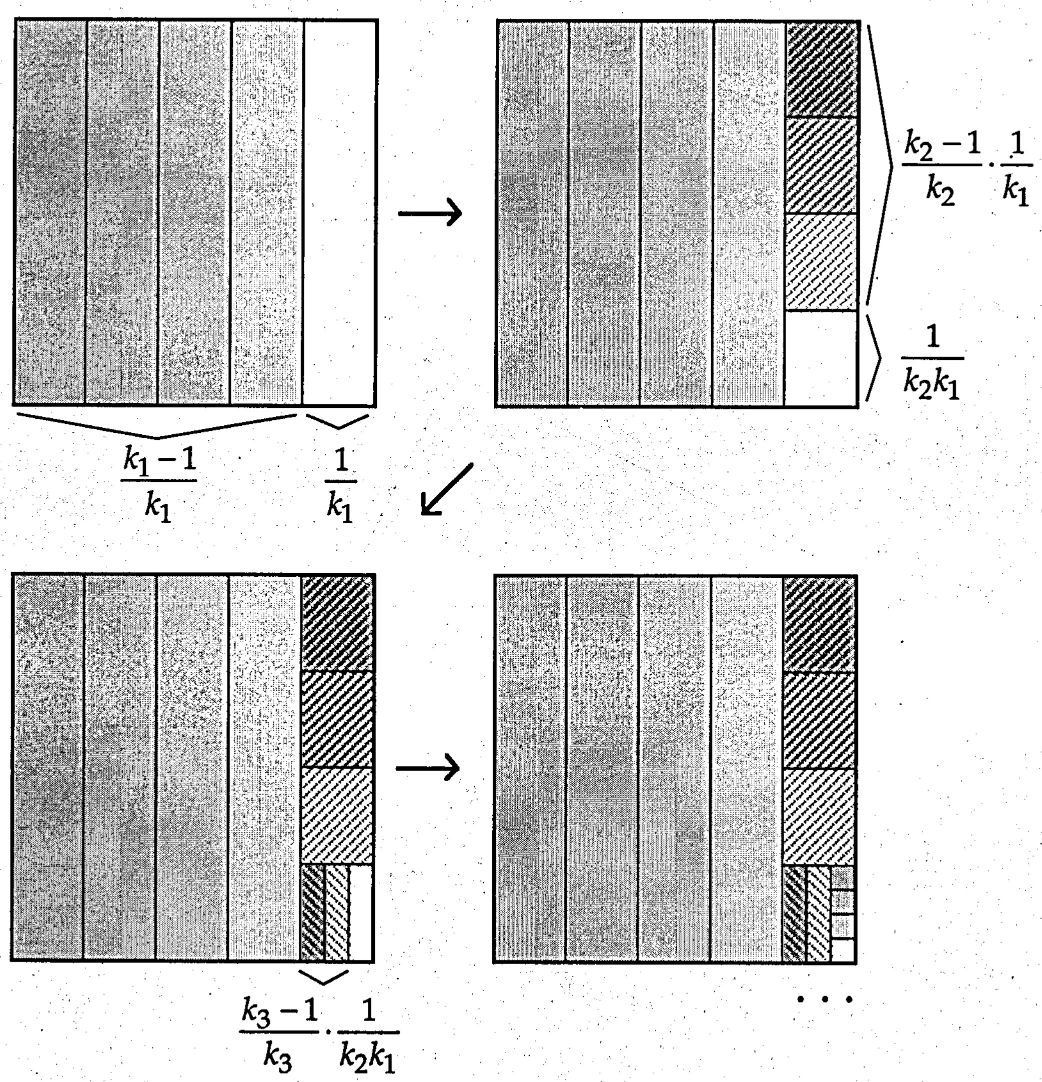
$$A_1=\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\frac{1}{32}-\frac{1}{64}+\cdots,$$
$$A_1=A_2=A_3,$$
$$A_1+A_2+A_3=1,$$
$$\therefore A_1=\frac{1}{3}.$$

—James O. Chilaka

A Generalized Geometric Series

Let $\{k_1, k_2, k_3, \cdots\}$ be a sequence of integers, each of which is at least 2. Then

$$\frac{k_1-1}{k_1}+\frac{k_2-1}{k_2k_1}+\frac{k_3-1}{k_3k_2k_1}+\cdots=1.$$

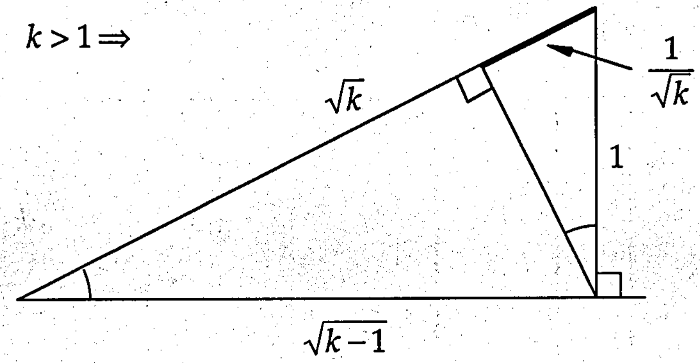


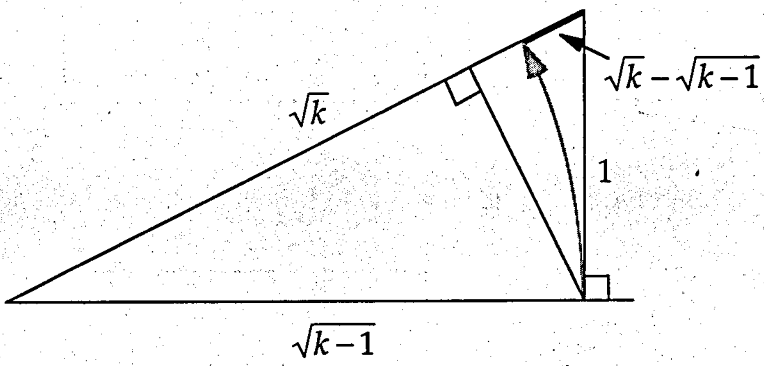
—John Mason

Divergence of a Series

$$n > 1 \Rightarrow \sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n}$$

$k > 1 \Rightarrow$





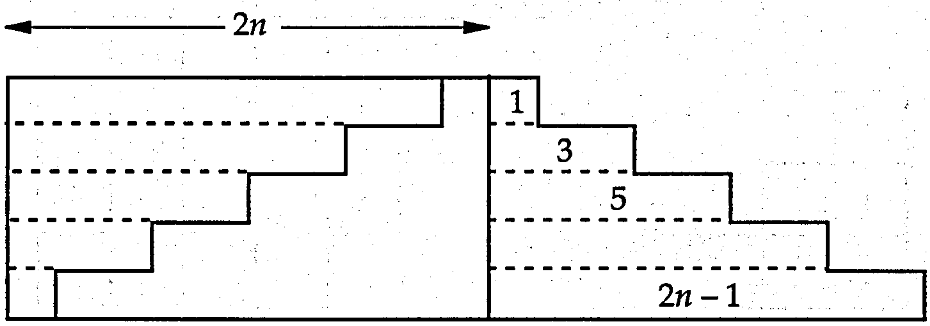
$$\frac{1}{\sqrt{k}} > \sqrt{k} - \sqrt{k-1}$$

$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \left(\sqrt{2}-1\right) + \left(\sqrt{3}-\sqrt{2}\right) + \cdots + \left(\sqrt{n}-\sqrt{n-1}\right)$$

$$\therefore 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$$

—Sidney H. Kung

Galileo's Ratios

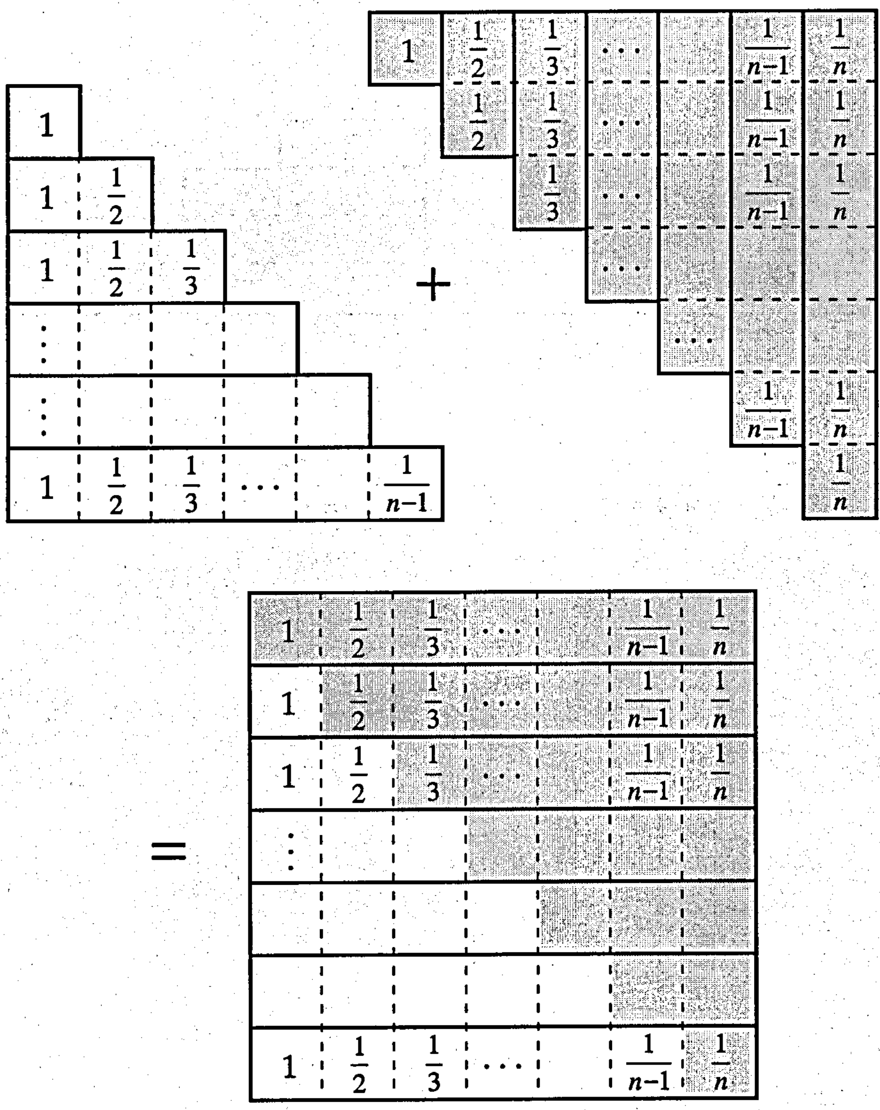


$$\frac{1}{3}=\frac{1+3}{5+7}=\frac{1+3+5}{7+9+11}=\cdots=\frac{1+3+5+\cdots+(2n-1)}{(2n+1)+(2n+3)+\cdots+(2n+2n-1)}$$

—Alfinio Flores

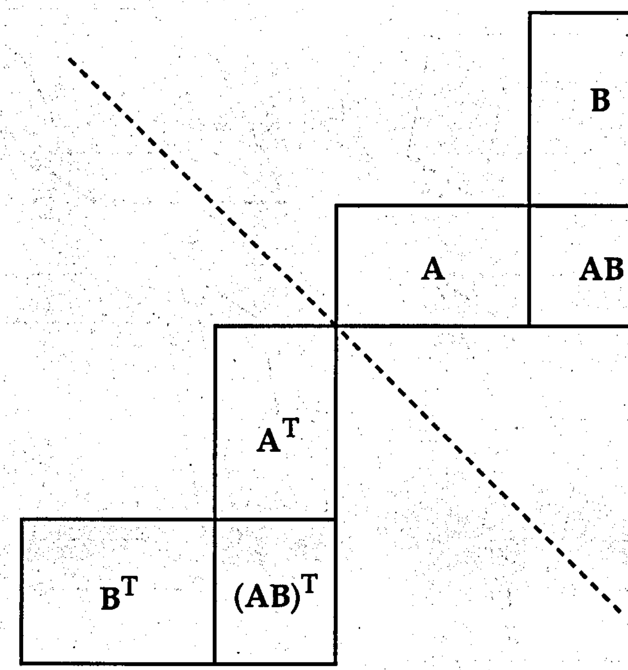
Sums of Harmonic Sums

$$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} \Rightarrow \sum_{k=1}^{n-1} H_k = nH_n - n$$



$$\sum_{k=1}^{n-1} H_k + n = nH_n$$

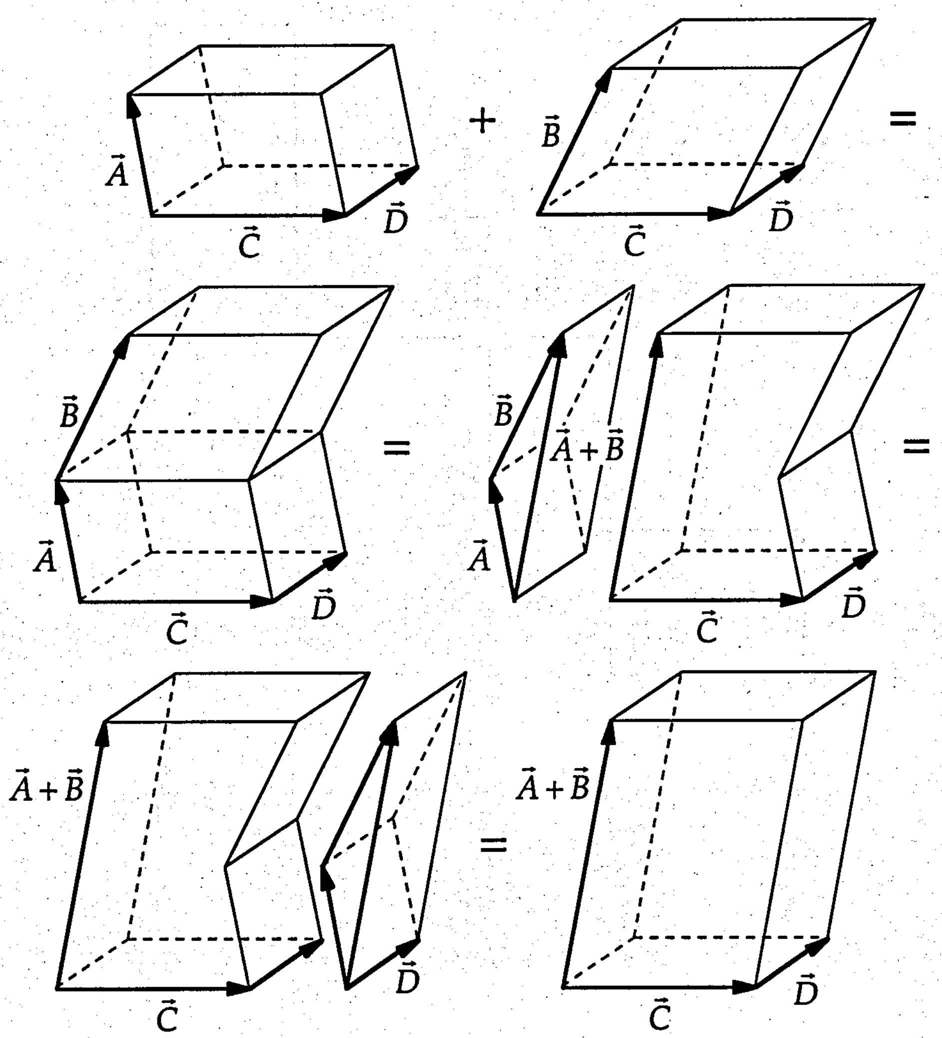
$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$, Where **A** and **B** are Matrices



—James G. Simmonds

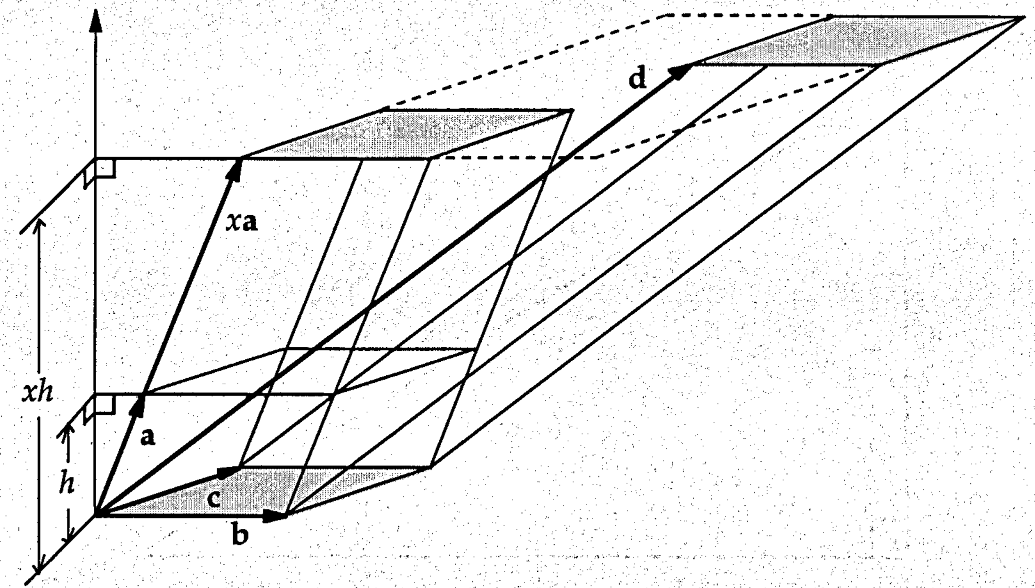
The Distributive Property of the Triple Scalar Product

$$\vec{A}\cdot\left(\vec{C}\times\vec{D}\right)+\vec{B}\cdot\left(\vec{C}\times\vec{D}\right)=\left(\vec{A}+\vec{B}\right)\cdot\left(\vec{C}\times\vec{D}\right)$$



—Constance C. Edwards
and Prashant S. Sansgiry

Cramer's Rule



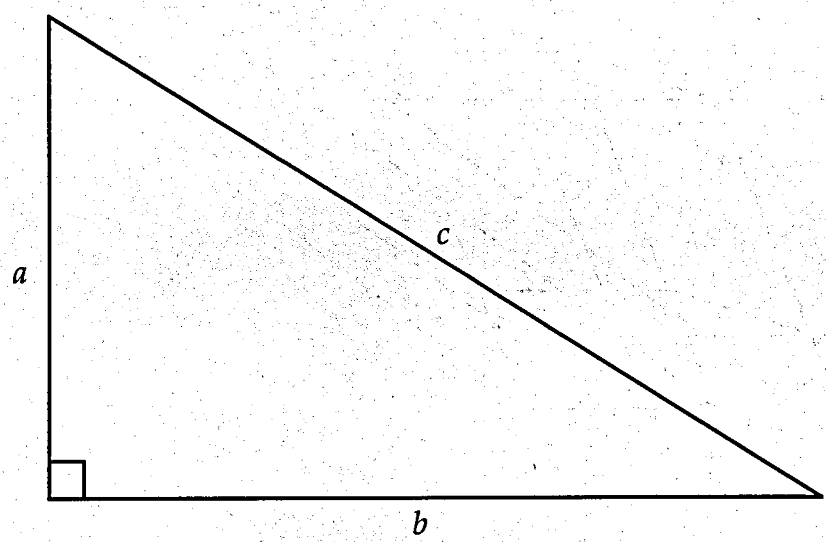
$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{d} \Rightarrow \det(\mathbf{d},\mathbf{b},\mathbf{c}) = \det(x\mathbf{a},\mathbf{b},\mathbf{c}) = x\det(\mathbf{a},\mathbf{b},\mathbf{c})$$

$$\therefore x = \frac{\det(\mathbf{d},\mathbf{b},\mathbf{c})}{\det(\mathbf{a},\mathbf{b},\mathbf{c})}$$

—*The Mathematics Initiative,*
Education Development Center

Parametric Representation of Primitive Pythagorean Triples

$$\frac{a}{2}, b, c \in \mathbf{Z}^+, \quad (a,b)=1$$



$$\frac{c+b}{a}=\frac{n}{m}, \ (n,m)=1 \Rightarrow \frac{c-b}{a}=\frac{m}{n},$$

$$\Rightarrow \frac{c}{a}=\frac{n^2+m^2}{2mn}, \ \frac{b}{a}=\frac{n^2-m^2}{2mn},$$

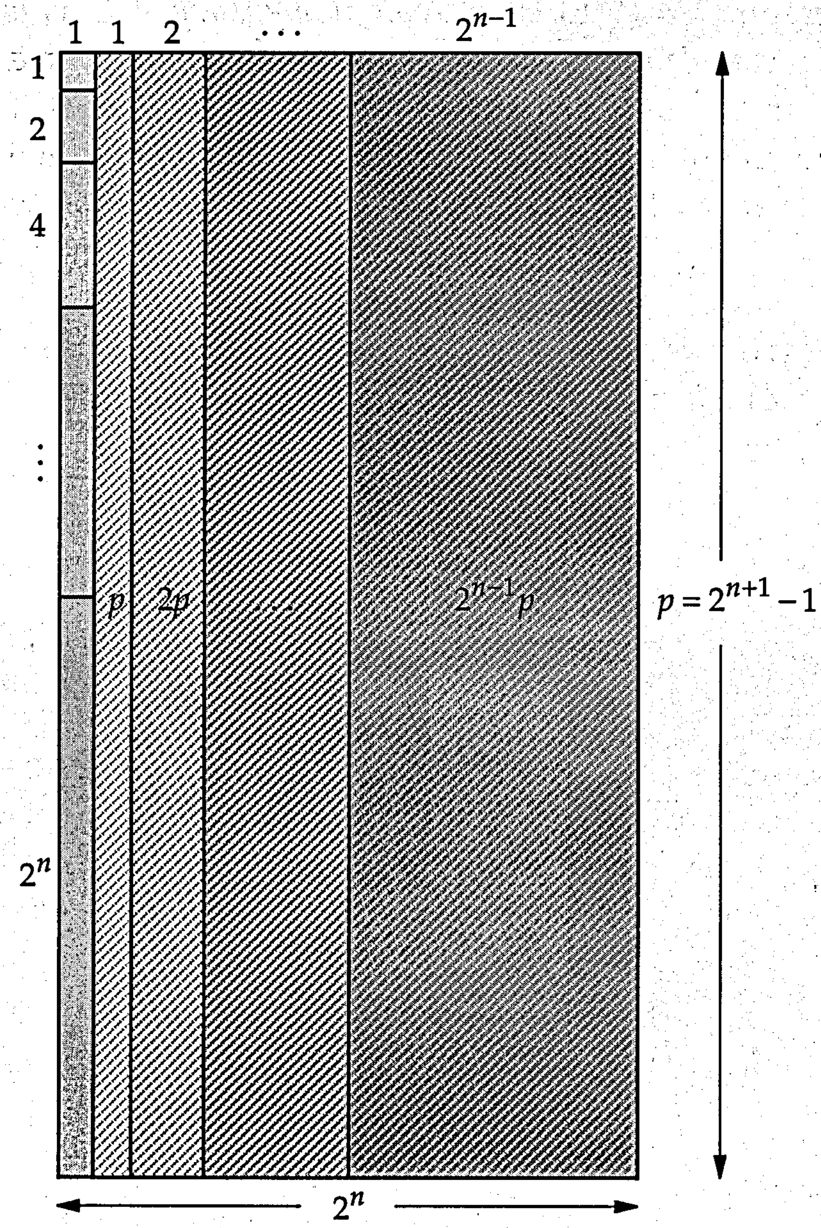
$$\Rightarrow n \not\equiv m \ (\mathrm{mod}\, 2).$$

$$\therefore (a.b,c)=(2mn, n^2-m^2, n^2+m^2).$$

—Raymond A. Beauregard
and E. R. Suryanarayan

On Perfect Numbers

$$p = 2^{n+1} - 1 \text{ prime} \quad \Rightarrow \quad N = 2^n p \text{ perfect}$$



$$1 + 2 + \cdots + 2^n + p + 2p + \cdots + 2^{n-1}p = 2^n p = N$$

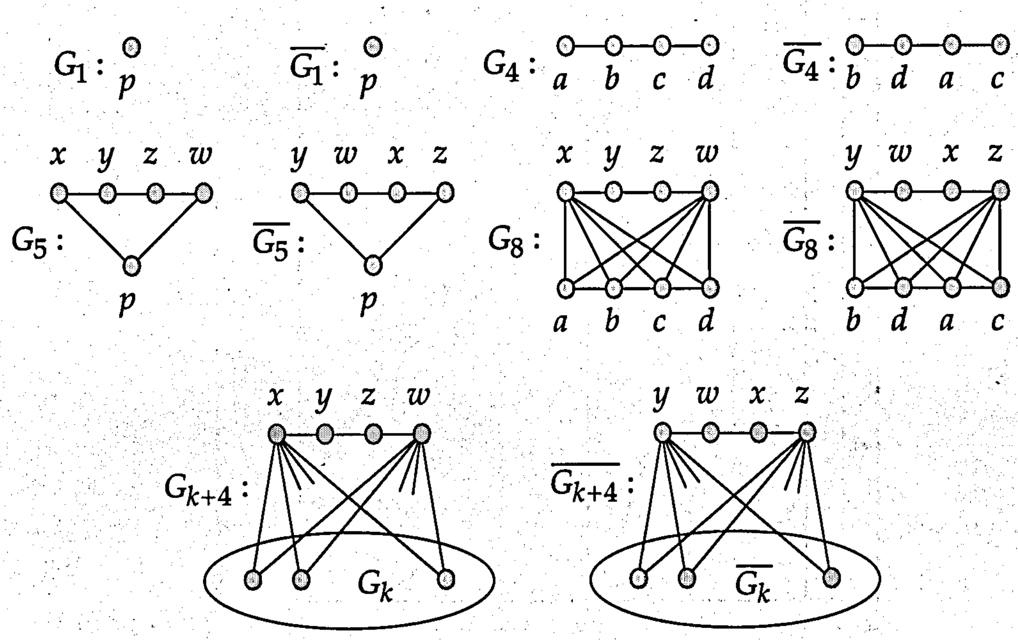
—Don Goldberg

Self-Complementary Graphs

A graph is *simple* if it contains no loops or multiple edges. A simple graph $G = (V,E)$ is *self-complementary* if G is isomorphic to its *complement* $\overline{G} = (V,\overline{E})$, where $\overline{E} = \{\{v,w\}: v,w \in V,\ v \neq w, \text{ and } \{v,w\} \notin E\}$. It is a standard exercise to show that if G is a self-complementary simple graph with n vertices, then $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$. A converse also holds, as we now show.

THEOREM: *If n is a positive integer and either $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$, then there exists a self-complementary simple graph G_n with n vertices.*

PROOF:



—Stephan C. Carlson

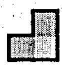
Tiling with Trominoes

A *tromino* is a plane figure composed of three squares: 

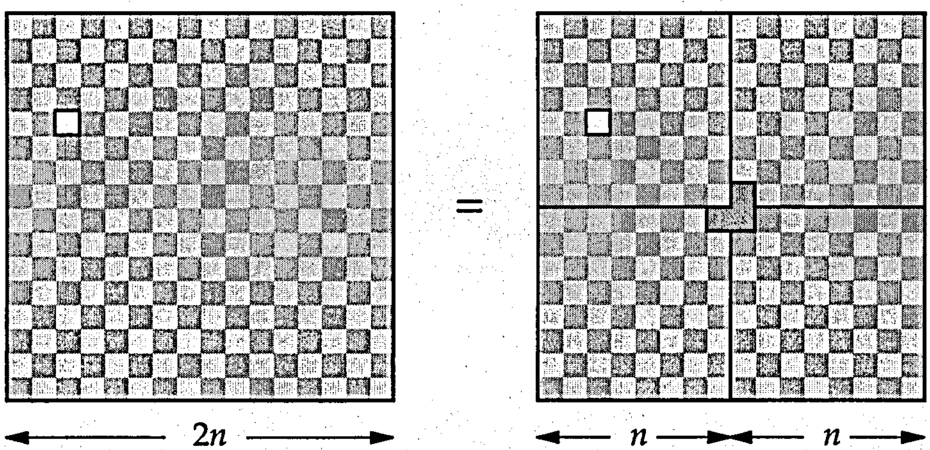
THEOREM: *If n is a power of two, then an $n \times n$ checkerboard with any one square removed can be tiled using trominoes.*

PROOF (by induction):

I.



II.



—Solomon W. Golomb

NOTE: Except when $n = 5$, an $n \times n$ checkerboard with any one square removed can be tiled with trominoes if and only if $n \not\equiv 0 \pmod 3$. See I-Ping Chu and Richard Johnsonbaugh, "Tiling deficient boards with trominoes," *Mathematics Magazine* 59 (1986) 34-40.

Sources

page	source

Geometry & Algebra

3 http://tug.org/applications/PSTricks/Tilings
4 *Mathematics Magazine,* vol. 71, no. 3 (June 1998), p. 170.
5 Howard Eves, *Great Moments in Mathematics (Before 1650),* The Mathematical Association of America, Washington, 1980, pp. 29-31.
6 Reprinted with permission from Elisha Scott Loomis, *The Pythagorean Proposition,* p. 112, copyright 1968 by the National Council of Teachers of Mathematics. All rights reserved.
7 *College Mathematics Journal,* vol. 27, no. 5 (Nov. 1996), p. 409.
8 *Mathematics Magazine,* vol. 72, no. 5 (Dec. 1999), p. 407.
9 *Mathematics Magazine,* vol. 71, no. 1 (Feb. 1998), p. 64.
10 *Mathematics Magazine,* vol. 70, no. 5 (Dec. 1997), p. 380; vol. 71, no. 3 (June 1998), p. 224.
11 *Mathematics Magazine,* vol. 71, no. 4 (Oct. 1998), p. 314.
12 http://www.cms.math.ca/CMS/Competitions/OMC/examt/english69.html
13 I. *Mathematics Magazine,* vol. 71, no. 3 (June 1998), p. 196.
II. Ross Honsberger, *Mathematical Morsels,* The Mathematical Association of America, Washington, 1978, pp. 27-28.
14 *Mathematics Magazine,* vol. 72, no. 4 (Oct. 1999), p. 317.
15 Written communication.
16 *Mathematics Magazine,* vol. 72, no. 2 (April 1999), p. 142.
17 Reprinted with permission from *The Mathematics Teacher* [vol. 85, no. 2 (Feb. 1992), front cover; vol. 86, no 3 (March 1993), p. 192], copyright 1992, 1993 by the National Council of Teachers of Mathematics. All rights reserved.
18 *American Mathematical Monthly,* vol. 93, no. 7 (Aug.-Sept. 1986), p. 572.
19 *College Mathematics Journal,* vol. 28, no. 3 (May 1997), p. 171.
20 *Mathematics and Computer Education,* vol. 33, no. 3 (Fall 1999), p. 282.
22 Ross Honsberger, *Mathematical Morsels,* The Mathematical Association of America, Washington, 1978, pp. 204-205.
24 *American Mathematical Monthly,* vol. 86, no. 9 (Nov. 1979), pp. 752, 755.

page	source

Geometry & Algebra (continued)

25 *American Mathematical Monthly,* vol. 71, no. 6 (June-July 1964), pp. 636-637.
26 Ross Honsberger, *Mathematical Gems III,* The Mathematical Association of America, Washington, 1985, p. 31.
27 *Mathematics Magazine,* vol. 67, no. 4 (Oct. 1994), p. 267.
28 *College Mathematics Journal,* vol. 25, no. 3 (May 1994), p. 211.
29 Ross Honsberger, *Mathematical Morsels,* The Mathematical Association of America, Washington, 1978, pp. 126-127.
30 *American Mathematical Monthly,* vol. 106, no. 9 (Nov. 1999), pp. 844-846.
31 *Mathematics Magazine,* vol. 71, no. 2 (April 1998), p. 141.
32 *Mathematics Magazine,* vol. 71, no. 5 (Dec. 1998), p. 377.
33 *College Mathematics Journal,* vol. 27, no. 1 (Jan. 1996), p. 32.
34 *College Mathematics Journal,* vol. 26, no. 2 (March 1995), p. 131.

Trigonometry, Calculus & Analytic Geometry

39 *Mathematics Magazine,* vol. 66, no. 2 (April 1993), p. 135.
40 Written communication.
41 *College Mathematics Journal,* vol. 26, no. 2 (March 1995), p. 145.
42 http://www.maa.org/pubs/mm_supplements/smiley/trigproofs.html
43 *Mathematics Magazine,* vol. 72, no. 5 (Dec. 1999), p. 366.
44 *College Mathematics Journal,* vol. 30, no. 3 (May 1999), p. 212.
45 *Mathematics Magazine,* vol. 72, no. 4 (Oct. 1999), p. 276.
46 *College Mathematics Journal,* vol. 30, no. 5 (Nov. 1999), p. 433; vol. 31, no. 2 (March 2000). pp. 145-146.
48 *Mathematics Magazine,* vol. 71, no. 5 (Dec. 1998), p. 385.
49 *College Mathematics Journal,* vol. 27, no. 2 (March 1996), p. 155.
50 *Mathematics Magazine,* vol. 69, no. 4 (Oct. 1996), p. 269.
51 *College Mathematics Journal,* vol. 29, no. 2 (March 1998), p. 157.
52 *Mathematics Magazine,* vol. 69, no. 4 (Oct. 1996), p. 278.
53 *College Mathematics Journal,* vol. 29, no. 2 (March 1998), p. 133.
54 *Mathematics Magazine,* vol. 71, no. 2 (April 1998), p. 130.
55 *College Mathematics Journal,* vol. 27, no. 2 (March 1996), p. 108.
56 *Mathematics Magazine,* vol. 71, no. 3 (June 1998), p. 207.
58 *College Mathematics Journal,* vol. 29, no. 2 (March 1998), p. 147.

page	source

Trigonometry, Calculus & Analytic Geometry (continued)

59 *College Mathematics Journal,* vol. 28, no. 3 (May 1997), p. 186.
60 *College Mathematics Journal,* vol. 29, no. 4 (Sept. 1998), p. 313.
61 *College Mathematics Journal,* vol. 27, no. 4 (Sept. 1996), p. 304.
62 http://www.iam.ubc.ca/~newbury/proofwowords/proofwowords.html
63 Written communication.
64 *College Mathematics Journal,* vol. 29, no. 1 (Jan. 1998), p. 17.
65 *Mathematics Magazine,* vol. 68, no. 3 (June 1995), p. 192.
66 *Mathematics Magazine,* vol.66, no. 2 (April 1993), p. 113.
67 *College Mathematics Journal,* vol. 30, no. 3 (May 1999), p. 212.

Inequalities

71 *Mathematics and Computer Education,* vol. 31, no. 2 (Spring 1997), p. 191.
72 *College Mathematics Journal,* vol. 26, no. 1 (Jan. 1995), p. 38.
73 *College Mathematics Journal,* vol. 25, no. 2 (March 1994), p. 98.
74 *Mathematics Magazine,* vol. 73, no. 2 (April 2000), p. 97.
75 *American Mathematical Monthly,* vol. 88, no. 3 (March 1981), p. 192.
76 *Mathematics Magazine,* vol. 68, no. 4 (Oct. 1995), p. 305.
77 *College Mathematics Journal,* vol. 26, no. 5 (Nov. 1995), p. 367, vol. 27, no. 2 (March 1996), p. 148.
78 *College Mathematics Journal,* vol. 25, no. 3 (May 1994), p. 192.
79 *Mathematics Magazine,* vol. 69, no. 2 (April 1996), p. 126.
80 *Mathematics Magazine,* vol. 37, no. 1 (Jan.-Feb. 1964), pp. 2-12.

Integer Sums

83 *College Mathematics Journal,* vol. 26, no. 3 (May 1995), p. 190.
84 *Mathematics Magazine,* vol. 70, no. 4 (Oct. 1997), p. 294.
86 *Mathematics Magazine,* vol. 70, no. 3 (June 1997), p. 212.
87 *Student,* vol. 3, no. 1 (March 1999), p. 43.
89 *College Mathematics Journal,* vol. 25, no. 2 (March 1994), p. 111.
90 *College Mathematics Journal,* vol. 25, no. 3 (May 1994), p. 246.
91 Written communication.
92 *Mathematics Magazine,* vol. 73, no. 1 (Feb. 2000), p. 59.

page	source

Integer Sums (continued)

93 *College Mathematics Journal*, vol. 29, no. 1 (Jan. 1998), p. 61.
94 *Mathematics Magazine*, vol. 71, no. 1 (Feb. 1998), p. 65.
95 *College Mathematics Journal*, vol. 27, no. 2 (March 1996), p. 118.
97 *Mathematics Magazine*, vol. 70, no. 1 (Feb. 1997), p. 46.
98 *Mathematics Magazine*, vol. 68, no. 4 (Oct. 1995), p. 284.
99 *Mathematics Magazine*, vol. 70, no. 2 (April 1997), p. 130.
101 *College Mathematics Journal*, vol. 28, no. 3 (May 1997), p. 197.
102 *Mathematics Magazine*, vol. 69, no. 2 (April 1996), p. 127.
104 *Mathematics and Computer Education*, vol. 33, no. 1 (Winter 1999), p. 62.
105 *Mathematics Magazine*, vol. 67, no. 5 (Dec. 1994), p. 365.
106 *Mathematics Magazine*, vol. 69, no. 1 (Feb. 1996), p. 63.
107 M. Bicknell & V. E. Hoggatt, Jr. (eds.), *A Primer for the Fibonacci Numbers*, The Fibonacci Association, San Jose, 1972, pp. 152-156.
108 *Mathematics and Computer Education*, vol. 31, no. 2 (Spring 1997), p. 190.

Infinite Series, Linear Algebra, and Other Topics

111 *Mathematics Magazine*, vol. 72, no. 1 (Feb. 1999), p. 63.
112 *Mathematics Magazine*, vol. 69, no. 5 (Dec. 1996), pp. 355-356.
113 *College Mathematics Journal*, vol. 26, no. 5 (Nov. 1995), p. 381.
114 *College Mathematics Journal*, vol. 26, no. 4 (Sept. 1995), p. 301.
115 *College Mathematics Journal*, vol. 29, no. 4 (Sept. 1998), p. 300.
117 *College Mathematics Journal*, vol. 26, no. 3 (May 1995), p. 250.
118 *Mathematics Magazine*, vol. 70, no. 2 (April 1997), p. 136.
119 *College Mathematics Journal*, vol. 28, no. 2 (March 1997), p. 118.
120 *Mathematics Magazine*, vol. 69, no. 3 (June 1996), p. 189.
121 Written communication.
122 *Mathematics Magazine*, vol. 73, no. 1 (Feb. 2000), p. 12.
123 *American Mathematical Monthly*, vol. 61, no. 10 (Dec. 1954), pp. 675-682.

Index of Names

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