What "Is" Mathematics?

In Memoriam of
Gian-Carlo Rota
Professor of Applied Mathematics and Philosophy
Massachusetts Institute of Technology
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It is an open secret among scientists that accurate predictions made on the basis of scientific laws are rare. Only yesterday, in the dark age of Carnap and Reichenbach, prediction was believed to be the fundamental feature of science. This unrealistic belief went hand in hand with a preposterous confusion, namely, the identification of all of science with physics. In that age of savage simplification, shining examples of prediction in science were the confirmations of the special theory of relativity, the explanation of the spectral lines of the hydrogen atom by quantum mechanics, and sundry other pillars of progress gleaned from the science of mechanics.

It took philosophers well over fifty years to carry out a reality check on their philosophy of science and to diagnose the normative disease that has plagued philosophy in this century. A casual inspection of any science other than physics or chemistry proves beyond reasonable doubt that scientific prediction in the sense of the positivists is at best a cruel joke. Zoology and cosmology, economics and evolutionary biology are only tangentially concerned with accurate prediction.

A description of the scientific enterprise pruned of normative presuppositions lies still in the future. Meanwhile, we may begin to chip away at the barriers that stand in the way. One such barrier is the systematic misuse of language by philosophers and logicians. Common words are rudely deprived of the multiple and contradictory meanings that they enjoy in ordinary language; after the straitjacket of a fixed meaning for every word is imposed, the door is shut to realistic description.

We have chosen the word "is" as paradigmatic of the constipation of meaning from which contemporary philosophy is suffering. We describe some of the multiple senses of the question: "What is mathematics?" when the question is asked in various circumstances.

The reigning orthodoxy of philosophy identifies the uses of "is" with the restricted uses of the word "is" in Fregean logic. Logic has achieved in this century a state of perfection that few mathematical theories have matched. However (or perhaps for this very reason), logical reasoning has become totally divorced from actual reasoning, the kind that is found in real life. In most worldly circumstances, logic shines by its absence. A compelling logical argument is the last weapon of the rhetorician, a recourse to be appealed to in desperate situations, when all else has failed.

It is no accident that substantial applications of Fregean logic are found daily in computer science. The crazies of the eighties, who pretended to simulate the mind with primitive computers, have succeeded, by a display of illiterate reductionism, in clearing up the abyss that separates human discourse from logical deduction.

It is thus no surprise to realize that the meanings of the word "is" prescribed by logicians are a lot closer to the ranting of Ayerian philosophy-fiction than to the richness of senses of the word "is" in everyday writing and conversation.

The accusation of being "illogical" may be leveled at us. Our retort will be a call to duty: realistic description is a paramount task of the philosopher. The first step in a philosophical description consists in admitting that the real is seldom rational and the rational is seldom real. An "Abgrund" separates "Verstand" from "Vernunft." Philosophical description must grapple with open-ended varieties of irreducible cases, with contradictory and ambiguous conclusions which Enlightenment Reason has ignored.

It is our contention that the word "is" in the question "What is mathematics?" does not have a classifiable set of meanings. This contention in no way implies that the word "is" is devoid of meaning. Quite the contrary: we are confronted with an "embarras de choix" among the meanings of "is."

What follows is a partial list of contexts in which the question "What is mathematics?" is found. The list is deliberately biased; it is meant to lead up to a conclusion decided upon in advance.

"IS" AS DICTIONARY DEFINITION

Literally, the question "What is mathematics?" calls for a "definition" of mathematics.

We have been trained to restrict the meaning of the word "definition" to the role of definition in axiomatic mathematical systems. This mathematical sense of the word "definition" will be henceforth disregarded. The senses of the word "definition" in ordinary discourse bear little relation to mathematical definition.

> ***The skeptical argument mistakenly concludes that no definition can be "given" in the dictionary, except in the eyes of the beholder.***

The word "definition" occurs in a great many vague and unclear senses, which it would be presumptuous to list. The most common, as well as the most ambiguous is the "definition" that we expect to find when we look up a word in a dictionary.

What is a dictionary definition? In what sense do dictionaries "define" words?

An old skeptical argument purports to prove that dictionary definition is impossible. It runs roughly as follows. Suppose you look up the meaning of word A. The dictionary explains the meaning of A in terms of words B, C and D, say. It may happen that B, C and D categorically specify A as the sole word satisfying certain conditions. But this happens very seldom. More frequently, the dictionary explanation of A in terms of B, C and D is likely to be a vague approximation to the meaning of A. The reader is asked to "get a feeling" for A by various tricks: the explanation of A in terms of B, C and D may be the description of a general class of which A is a member, or it may be a list of likenesses, of comparisons with other objects that are meant to be "like" A; one reads various indirect hints to the meaning to A. What cannot be given in a dictionary is "the meaning" of A.

This frustrating remark by no means implies that the reader will miss the meaning of A when looking up A. The reader is expected to grasp the meaning of A by letting his imagination roam "beyond" the various statements in the dictionary that are meant to "lead up to" the meaning of A. The meaning of A can be grasped only when one looks "away" from the dictionary explanations "towards" some other sense that is not given there, but which the dictionary explanations "point to." No amount of explanation can make sure that the reader will take the leap that will disclose the meaning of A.

The skeptical argument mistakenly concludes that no definition can be "given" in the dictionary, except in the eyes of the beholder.

The preceding argument stands in contrast with what actually happens when someone looks up a word in the dictionary. In point of fact, people do find the meaning of words in the dictionary. If we look up the word "jaguar," we will get an adequate idea of a jaguar, even if we have never seen a jaguar or know nothing about jaguars. When I look up the word "chair," I get a pretty good idea of what a chair is, even if I do not quite grasp the full meaning of what I have "read" until I become familiar with actual chairs.

Dictionaries of synonyms are a further confirmation of the same phenomenon. One learns the meaning of a new word from approximate explanations, by a process that cannot be rationally accounted for. When we look up the synonyms of a known word, we are searching for some word that we may never have seen. We approximately guess the meaning of the synonyms, even though these meanings are nowhere given, and we select an appropriate synonym which we may never have previously seen or used.

The non-rational grasping of the meaning of a word from approximate explanations is an instance of the phenomenon of the "copula," of the function of "is" in "A is B." What matters here is that the "is" acts as a copula only for certain A's and B's, like "jaguar" and "chair." Certain other words pose a different problem of "is" that is not subsumed in the "copula" sense of "is." One such word is "mathematics." Any "mathematics is..." sentence given in response to the question "What is mathematics?" will be evasive. No sensible dictionary definition of "mathematics" can be given.

"IS" AS INVITATION

The situation is different when we look up the word "mathematics" in an encyclopedia rather than in a dictionary. In an encyclopedia we find summaries of entire mathematical fields, as well as a bird's eye view of various branches of mathematics and an ample bibliography that will guide us to learning mathematics.

Is the description of "mathematics" in an encyclopedia an adequate answer to the question "What is mathematics?" unlike the dictionary? Clearly not. The explanation of "mathematics" that we find in an encyclopedia skirts the question by referring us, after an enticing preamble, to technical expositions and classical treatises.

Both the dictionary "is" and the encyclopedia "is" are motivated by the widely felt need of explaining esoteric words in exoteric language. This need roughly dates back to the Renaissance, when the first dictionaries (in the contemporary sense of the term) were compiled. Throughout history, notably in the Middle Ages, no need for exoteric expositions was felt. Explanations (often labeled "definitions") were an internal affair for specialists, from which the public was excluded. The scholastic definition of "Deus" as "eng perfect-issimum" was meant to be shared by philosophers and theologians only. Uttering such a statement in the course a Sunday sermon at Mass might have led to an accusation of heresy.

The Renaissance-Enlightenment notion of definition as exoteric explanation is motivated by the democratic ideal of a universal culture. Such a laudable objective does not make exoteric explanation any easier. Fortunately, an exoteric explanation of mathematics is seldom what the questioner expects when posing the question "What is mathematics?". Let us see.

"IS" AS COPOUT

The question "What is mathematics?" is often asked when the questioner has little or no acquaintance with mathematics, and wants to discharge his or her duty to learn something about mathematics, hoping for a short answer.

The question "What is mathematics?", asked to a mathematician by a person ignorant of mathematics, makes mathematicians uneasy. The mathematician senses dishonesty in the abruptness of the question. The questioner believes that an answer can be given, similar to the answers one can give to questions like "What is boeuf bourguignon?", "What is yellow fever?" or "What are Magli shoes?".

The questioner does not want to learn any mathematics when he asks the question "What is mathematics?". The opposite is true: the questioner wants to rid himself of the need of learning any mathematics whatsoever. He wants to add to his conversational repertoire some brilliant answer that will permanently excuse him from any further dealings with the subject.

One cannot escape the duty of giving a nutshell answer to the question "What is mathematics?", despite the dishonesty of all short answers. Non-mathematicians need to have some idea of what mathematics "is" without having to study mathematics. They are dealing with mathematics as outsiders, but their dealings will affect the future of mathematics: mathematics requirements for schools must be determined by professional educators; mathematical proficiency among employees in a firm has to be gauged. Worst of all, the allocation of research funds for mathematics is made by individuals who have at best a fleeting acquaintance with the subject. Mathematics, like all intellectual disciplines, is not economically self-sustaining , and since the beginnings of civilization mathematicians have depended for their survival on the largesse of society or of a few wealthy individuals. Mathematicians, like philosophers and artists, are "kept" persons. In return, the public expects mathematicians to make the results of their work accessible to cultivated persons who may have a passing interest in mathematics, or who deal with the political and economic problems of mathematicians.

We will leave to another occasion the tragedy that has resulted from the mathematicians' failure, going all the way back to Pythagoras, of giving exoteric accounts of their field that the public could appreciate.

An accessible and short answer to the question "What is mathematics?" may be difficult to give, it may turn out to be dishonest and inadequate, but the mathematicians' failure to provide such an answer has been a costly mistake.

"IS" AS ESCAPE

Students confronted with the task of learning a mathematical theory rarely feel the need to ask the preliminary question "What is mathematics?". They are more likely to ask specific questions, such as "What is topology?", "What is the Riemann hypothesis?", "What is a random variable?".

> *Are we to infer that no answer to the question "What is mathematics?" can ever be given?*

Suppose nevertheless, by way of thought experiment, that a student of mathematics were to ask such a question, on the basis of his claim that an answer to the question is a condition to be met preliminary to his getting down to serious study.

It is likely that a teacher hearing such a question from a student would give the student a strange look. The teacher would be put on guard: is the student unfamiliar with grade-school mathematics? is the student afraid of learning mathematics? does the student believe that an authorization is to be granted before undertaking the study of mathematics? is the student afraid of mathematics? does this student require medical attention?

In each of these instances, the teacher will not hazard an answer to the question. Most likely, the teacher may whisper to the student a few soothing words, not in the least meant to provide any explanation of what mathematics is, but rather meant to allay the anxieties that the student's question betrays.

"IS" AS SUMMING UP

Some mathematicians who are reaching the end of their careers (Poincaré, Hadamard, Weyl), feel the need to answer the question "What is mathematics?" as a prop to their fading hold on the subject, much as they might feel the need to write an autobiography. In these circumstances, the question "What is mathematics?" is an excuse for excursions into the history and philosophy of mathematics. The essays written in answer to this rhetorically posed question will deal with the "nature," the "structure," the "standing" of mathematics. The "is" is once more skirted by being turned into an "about," into discussions about the mathematics of the time, about future directions of mathematics, about relationships among various fields of mathematics.

"IS" AS IMPOSSIBLITY

We have argued that no answer to the question "What is mathematics?" can be given in the form "mathematics is..." by examining some contexts in which the question is asked. In none of the instances considered can the question be given an answer of the form "mathematics is..." In the first instance an answer of the form "mathematics is..." may be read in a dictionary, but such an answer is not taken seriously.

Are we to infer that no answer to the question "What is mathematics?" can ever be given?

Let us call a word X a "pre-ontological term" whenever no adequate answer to the question "What is X?" can be given in the form "X is..." The preceding examples suggest that "mathematics" is a pre-ontological term.

Most words of common usage are not pre-ontological terms. For instance the word "chair" is not a pre-ontological term, since we can answer the question "What is a chair?" by sentences of the form "a chair is..." An adequate such set of sentences will provide a description of chairs that is good enough for most purposes, even though no set of sentences may succeed in "defining" the word "chair" in the logical sense. We use the word "item" to denote any word X for which an adequate (though not necessarily logical) answer can be given to the question "What is X?" in the form "X is..." "Chair," "triangle," "jaguar" are items. Our claim is that there are pre-ontological terms, and pre-ontological terms are not items.

The philosophical literature is rich in pre-ontological terms: "time," "world" and "nothing" are three pre-ontological terms that have been studied in the phenomenological literature. The question "What is time?" has been deemed unanswerable by philosophers since St. Augustine. The question "What is 'nothing'?" is obviously intractable. In Chapter 3 of *Sein und Zeit*, Heidegger argues that "world" is a pre-ontological term.

The limitations of the language by which we describe and define items, a language made up of "A is B"-type sentences, stand in the way of describing (let alone defining) pre-ontological terms. One is forced to choose between two alternatives: either to decide that no sense can be made of sentences of the form "X is..." whenever X is a pre-ontological term, or else to find a sense of the word "is" that is distinct from the "is" as copula in the language of items.

The first alternative was followed in phenomenology, by an argument—the joint work of several authors—that we will try to sketch.

What "common" features of the words "mathematics," "time," "nothing" and "world" lead us to classify these words under the heading of pre-ontological terms?

The word "is" used as copula in "A is B" presupposes a context of sense-making. A can only be B within a background of unthematized features that are ordinarily passed over in silence. More formally: the "is" of "A is B" presupposes a context within which the "is" can "be." For example, "chair" presupposes a context of everydayness in which chairs are useful. No item can "be" without some such background context. "To be" is "to be-in-a-context." We read, pronounce, deal with the sentence "A is B" while pretending that the meaning of the sentence is to be found "in" the sentence itself, independently of any contextual background. The canons of logic foster the pretense of a decontextualized meaning of "A is B"; the cliché sentences given as examples in logic textbooks are carefully cleansed of contextual references. One can hardly imagine any such sentences ("the snow is white") ever used in daily conversation. But whenever "A is B" is used meaningfully, i.e., contextually, an unthematized background context can always be brought to the fore.

The "is" in "A is B" purports to explain A in terms of B. Such an explanation is made possible by a multilayered twine of contextual and intercontextual senses that link A to B. Without such an underlying contextual/intercontextual twine, no sense can be made of "A is B."

The "is" of "A is B" is meaningful if both A and B are items, i.e., whenever both A and B are ensconced in a common context. However, the statement "A is B" becomes problematic when either A or B are pre-ontological terms. Pre-ontological terms are not items, but conditions of possibility of the contextuality that allows items to "be." In plain words: no sentence of the form "time is..." can make sense, because "time" is not an entity of any kind, but a condition of possibility of all entities.

However, the impossibility of making sense of any "time is..." sentence does not deliver us from the problem of understanding the pre-ontological phenomenon of time. Rather, it points to the need for a language other than the language of items that will be suitable for the inquiry into the sense of time.

No "definition" of the term "mathematics" can describe that particular context that we call mathematics. Mathematics is not an item that certain contexts share. Mathematics is the condition of possibility of mathematical contexts. We cannot explain what mathematics "is" by sentences of the form "A is B," where A and B are items, because mathematics "is" no-thing.

The word "is" is misused when we try to explain what mathematics "is" in the language of contextual items. Questions like "What is mathematics?", "What is time?", "What is the world?" are misleading. Mathematics, time and world are not items, and hence it makes no sense to ask what they "are."

"IS" AS A WONDER

Are we to conclude that the question "What is mathematics?" should be dismissed as meaningless? Such a conclusion would be strikingly similar to the anathemas of the positivists, always ready to liquidate as "meaningless" any question beyond the reach of their narrow vocabularies. Besides, such a conclusion would bring back the specter of normative philosophy from which we have proudly distanced ourselves.

The question "What is mathematics?" is not always asked by way of a copout, as in the examples above. The question "What is mathematics?" is sometimes posed, both by the student and by the mature mathematician, to express a feeling of wonder, to signify the estrangement that possesses us at times, the same estrangement that is felt in the contemplation of the starry sky and the moral law, described by Kant at the beginning of his "Critique of practical reason." This feeling of estranged wonder is the opening to philosophical inquiry, as Aristotle was first to note. The question "What is mathematics?" may express the feeling of the wonder at the contemplation of the awesome edifice of mathematics.

The feeling of wonder that is sometimes expressed by the question "What is mathematics?" is not likely to be an "answer" to the question "What is mathematics?" It will be the start of a philosophical journey that will eventually disclose of the "conditions of possibility" of mathematics. The disclosure of such conditions of possibility is the "answer" to the question.

Sadly, philosophers have neglected the task of giving a rigorous formulation of the method of reasoning that leads to the disclosure of conditions of possibility. If the day ever comes when the "logic" of conditions of possibility, i.e., philosophy, is developed with the standards of rigor that have been set by Fregean logic, then an "answer" to the question "What is mathematics?" will be possible in the form "mathematics is ..."