Emil Saucan

Euler's Theorem as the Path towards Geometry

The course in Mathematics for Architecture Students at the Technion–Israel Institute of Technology needed to encapsulate as much formative knowledge as possible. Above and beyond the absolute importance of Euler's Formula relative to the corpus of classical mathematics and the role it played in its development (as in Betti numbers and Homology in general on one hand and the Global Gauss-Bonnet Theorem on the other hand), its simplicity, yet potency (in the sense of representing a spring board, an opening towards a variety of subjects belonging to the fields of Topology and Geometry) recommend Euler's Theorem as natural candidate for a cornerstone, a red thread running through and directing the whole course

1 Introduction

The roots of this article reside as much in curricular pragmatism as in programmatic, ideologic convictions. By curricular pragmatism we mean the anankian[1] drive for one single course in Mathematics for Architecture Students at the Technion–Israel Institute of Technology, course that would encapsulate as much formative knowledge as deemed feasible. The need for an "Object Oriented", fast approach, compact course, arises from curricular constraints: once the second half of a tandem of three hours per week, semestrial courses (the first one comprised elements of Matrix Algebra and Introduction to Calculus), it was reduced to a single semestrial course, of two weekly hours (the Algebra-Calculus half being abandoned completely). Moreover, this reduction in scope was accompanied by an augmentation of the syllabus: while the previous geometric course comprised only symmetry (albeit treated in some detail[2]), the new course was envisioned as a comprehensive introduction to incidence and symmetry of geometrical objects, an approach that is commonly hold to represents the cornerstone, the main goal of a Course of this type (the chosen basic reference text is [Baglivo and Graver 1983]). Thus, above and beyond the absolute importance of Euler's Formula relative to the corpus of classical mathematics and the role it played in its development (Betti numbers[3] and Homology in general, on one hand and the Global Gauss-Bonnet Theorem[4] on the other hand, being, not the least of its offshoots), its simplicity, yet potency (in the sense of representing a spring board, an opening towards a variety of subjects belonging to the fields of Topology and Geometry) recommend Euler's Theorem as natural candidate for a cornerstone, a red thread running through and directing the whole course.

Since a second purpose of any such course is to help develop geometric intuition and spatial imaginative powers, Euler's Theorem introduces one effortlessly to the realms of geometric creativity, by its natural generalizations into two directions: (a) high genus and non-orientable surfaces such as the Klein Bottle and the projective plane (thus representing an excursum in non-Euclidian geometries and also into patterns and tilings); and (b) Star and Uniform Polyhedra. (We shall indicate how and where these objects and ideas arise.)

Moreover, this approach allows for a perfect integration between the metric and topological aspects, thus presenting the dialectics of Mathematics, represented by the computational-arithmetic/figurative-geometric dichotomy.[5]

We shall show in what manner the path outlined herein facilitates a potent yet flexible curriculum: one can put more accent on graph theory or, alternatively, on "rigid" geometry, on incidence and topology or rather upon trigonometric computations. Through graphs one attains such applications as: organization graph of architectural structures (introducing duality and planarity), traversability, routing, connectedness, classification of architectural plans. Even the very approach to the proofs shines light on different aspects of geometry.

Upon these manifold proofs and the various, kaleidoscopic facets of geometry they illuminate, we shall dwell (alas, briefly!) in the following section.

2 The mathematics

The five Platonic solids, their role in the development of geometry and philosophical ideas of classical Greece (see [Crowe 1969, 2]) as well as and the inevitable Keplerian cosmogonical vision[6] with its sense of wonder (see [Crowe 1969, 4]) provide the best introduction into the realm of polyhedra. Therefore the proof of the existence of only five Platonic solids provides an excellent motivation for introducing Euler's Theorem.[7]

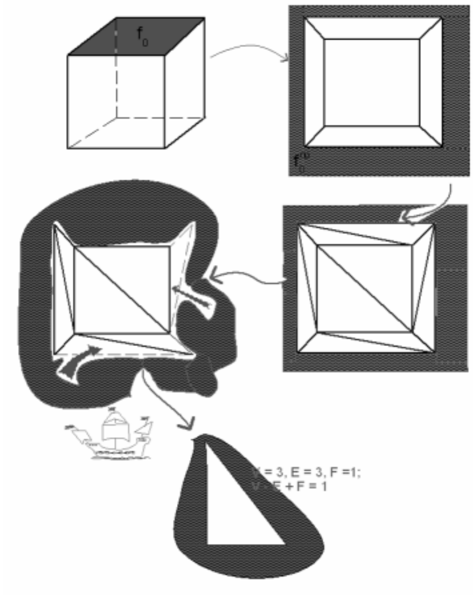

Fig. 1. The First Proof

Our proof of choice, the first one we present to our students, is basically Euler's original one, based upon first triangulating the polyhedron, and then one by one removing the resulting triangles, while showing that the number $V - E + F$ remains invariant during this process. As a special didactic trick, we present Gamow's variant of this proof, (see [Gamow 1988]).[8] In Gamow's presentation the polyhedron's edges are viewed as dams, the polyhedron as an isle, and the exterior of the resulting map as the sea. As a special flourish, we present this as a last defence against a Spanish Armada invading the Netherlands, sometime in a mythical sixteenth century (see Fig. 1). The advantage of this proof resides in it simplicity and in the fact that it introduces the extremely important notion of triangulation. But it provides us with even more depth: since an essential step in the proof is the removal of the upper face f_0 (see Fig. 1) and the projection of the reminder of the polyhedron on the plane,[9] this allows us to discuss the stereographic projection.

(Considering projections can also lead to Steiner's proof of Euler's Theorem[10]; see [Sommerville 1942, 142]). Moreover, faces as homeomorphic images of the disk and the topological concept of map (as opposed to that of a mere graph embedding) come under study naturally. From this point on, the possibilities are practically unlimited. The most direct road leads to counting the regular tilings of the plane and to that of Archimedian solids and semiregular tilings of the plane.

But the main advantage lies, first and foremost, in the fact that one can introduce the notion of graphs in a light, natural manner, and with them a variety of problems of great significance in the theoretical setting and of vast practical importance, such as planarity and duality (via "the five brothers problem" [Baglivo and Graver 1988] and the "gas-light-water graph" [Baglivo and Graver 1988; Tietze, 1965] and thus to the theorems of Kuratowsky and Whitney [Baglivo and Graver 1988; Tietze, 1965]. Trees are the simplest truly interesting graphs and they provide us with a third proof of Euler's Theorem: Von Staudt's proof [11] based upon spanning trees. Paradoxically, the benefit of this proof is that it does not extend to surfaces of positive genus—thus allowing yet a different insight into the topology of surfaces. Applications of Euler's Theorem, such as Kuratowski's theorem are now natural—through the inequality $E < 3V - 6$ ("The little inequality that could"). Its dual inequality $E < 3F - 6$ provides us with an easy proof of the six color theorem (and with an excuse to wander into a discussion of the four color theorem).

Planarity and maps conduct us immediately to consider surfaces other than the plane or the sphere, such as the torus and compact, orientable surfaces of higher genus, and also non-orientable surfaces, in particular the Möbius strip, the Klein bottle and the projective plane (see Fig. 2).

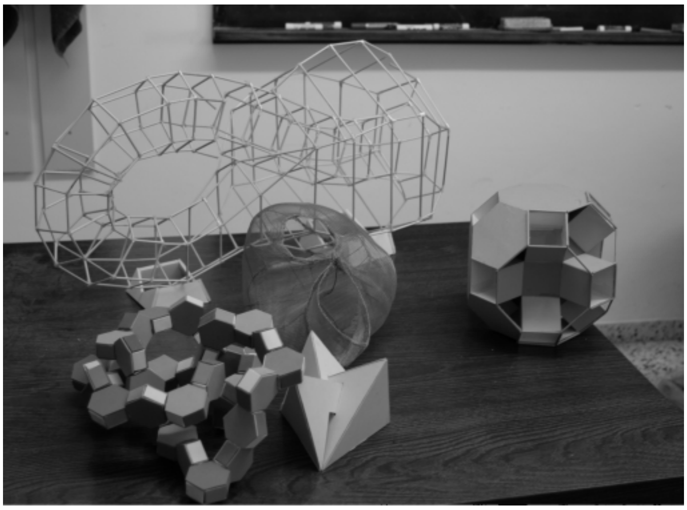

Fig. 2. Topological Models

Also, as an immediate—yet somewhat collateral—development, maximal planar graphs and fundamental architectural arrangements [Baglivo and Graver 1983, 115-119] ensue. Yet another proof is needed if one considers further generalizations, such as star polyhedra and the even more general uniform polyhedra[12] (see Fig. 3).

For these one has to appeal to the spherical area proof (or Legendre's Proof) [Coxeter 1963].

Fig. 3. Star and Uniform Polyhedra

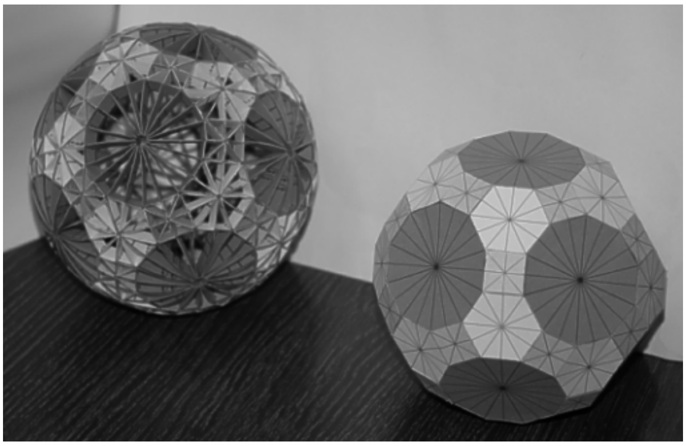

Fig. 4. An Archimedian Solid and its Spherical Counterpart

While a formal treatment of surface coverings is, of course, out of the course's scope, its rudiments are highly intuitive. Moreover, a much more technical instrument now becomes tangible: a numerical solution to compute the elements (sides, dihedral angles, radii, etc.) of a Uniform Polyhedron. The method above—based upon a many variable adaptation of Newton's Tangent Method for solving equations—is the one developed in [Har'el 1993]. Even if in the beginning the students tend to oppose (at an instinctive, self-preserving level) the meanders of its computation, in the end, the concreteness and clarity of the numerical results obtained rewards them with a better insight and understanding of the objects they studied until this point only on a descriptive, visual way. This is an intrinsic result sprouting from the very depths of mathematical thought and understanding (but we shall resume this discussion again in the last section).

3 Didactical aspects

At this stage is only natural that one should try and estimate the didactical/methodological benefits of this course structure. We do believe this approach unifies the different aspects/parts of the course, giving a coherent perception that expunges false compartmentalization (sadly so often permeating mathematics curricula and, indeed, education). Following the natural road opened by Euler's Theorem allows the student to recognize geometry as the "art of posing questions" ([Gromov 1998])—as opposed to the mere ability to solve standard, technical exercises, the revered torturer of our high school days—and not to view it as a perpetual unpalatable compromise between dual, mutually exclusive entities—figurative expression and computational technique—but rather to perceive its dialectic nature as the interplay between two aspects, two functions of the mind, since "Our brain has two halves: one is responsible for the multiplication of polynomials and languages, and the other half is responsible for the orientation of figures in space and all the things important in real life. Mathematics is Geometry when you have to use both halves" [Arnold 1997].

Moreover, the focus on polyhedra facilitates the use of constructive projects as marking tool, thus addressing the demiurgical skills of the students, and adding yet another unifying, summarizing tool at the end of the course. Indeed, architecture students have not only the ability and the habit to express themselves in a constructive manner, they possess the propensity, the proclivity, for this Renaissance type of expression: material and spiritual, combined (see Fig. 4).

Fig. 5. A Cornucopia of Models

Also, it is this author's firm belief that "People are much smarter when they can use their full intellect" ([Thuston, 1990]). Further, creative freedom and trust are stronger moving forces and better guaranties of novel, original ideas (for the final projects) then some constrictive exam frame. Trigonometric equations may be tedious and boring, but the become your equations when they help finalize a project (see, for example the two origami polyhedra of Fig. 6).

Fig. 6. Two Origami Polyhedra

4 Beyond Euler's theorem

When the matter is so vast and so generous and the audience is so artistically inclined and technically able, as in our context, one should not restrict himself to the mere mathematical and didactical considerations, but should rather ask himself what is the deepest possible impact of his course, and what is the intellectual message it should convey.

It is stated in [Consiglieri and Consiglieri 2003] that "...mathematics does not lead to emotional forms but abstract ones; that responsibility belongs to aesthetics". This statement may conform to the common feeling of practicing artists. Nevertheless, it contradicts the very cultural tradition that resides at the inner core of the choice of any course in mathematics for architects, the fact that "Classical mathematics is a quest for structural harmony" [Gromov 1998]. As yet another of the titans of contemporary mathematics confesses: "Mathematics has a remarkable beauty, power and coherence, more than we could have ever expected" and "Mathematics is like a flight of fancy " [Thurston 1990]. And again, in Freudenthal's words: "... mathematics is an interplay of content and form",[13] yet couldn't this serve as a concise, functional definition of art?

And even if we restrict ourselves to a more mundane, practical level: if a course offers its listeners a plethora of examples, a gallery of fantastic forms, to fertilize, to help germinate and to serve as nutrient for growth of pure art, wouldn't it have served his purpose? With this question that expresses a hope, a belief, in a positive answer, we conclude our essay.

Acknowledgments

The author would like to express his gratitude to Prof. Tzvi Har'el for introducing him in the fascinating domain of architectural mathematics. Many thanks are also due to Eli Appelboim for his friendly and sustained help and to the anonymous referee for his/hers kind comments and thoughtful suggestions.

Notes

1. From "Ananke", Greek goddess, personification of unalterable necessity, compulsion, or of the force of destiny. Also "anankastic syndrome"—perfectionistic traits expressed as meticulous conscientiousness, preoccupation with rules and procedures, rigidity of behaviour.
2. In an even earlier incarnation it also comprised elements of Differential Geometry of Surfaces—see [Har'el 1985].
3. This notion requires some mathematical formalism: Let X be a topological space. The k-th Betti number β_k of X is defined as the rank of the k-th homology group of X: $\beta_k = \mathrm{rank} H_k$.

4. Let S be a compact surface. Then $\iint_S K dA = 2\pi\chi(S)$, where K represents the Gauss curvature and $\chi(S)$ is the Euler characteristic of S.
5. See [Arnold 1997] and Section 3 below.
6. Of nested cosmological regular polyhedra.
7. The geometric, angle based proof, is also presented, but more like an afterthought, an instructive variation.
8. A proof that made this author choose mathematics as his profession—this, and Cantor's Diagonal Proof.
9. Thus reducing the problem to proving that, for the new map $V - E + F = 1$.
10. The basic idea of Steiner (and Lhuilier's) proof is to project the polyhedron orthogonally on a plane, obtaining a polygon covered twice by a set of polygons, then express the sum of the angles of these polygons in terms of *V*, *E*, *F*.
11. The main steps of Von Staudt's proof are as follows: Build the spanning tree of the vertex set—the number of its edges will be $E_1 = V-1$. Construct also the dual tree (of the face set)—the number of edges will be $E_2 = F - 1$. Since the trees are disjoint we have: $E = E_1 + E_2 = V + F - 2$, i.e., $V-E+F = 2$.
12. And, of course, Regular and Uniform Polyhedra direct us to the study of Symmetry Groups.
13. Indeed, mathematicians refer to their object of study (or should we say "passion"?!) in aesthetic terms: "What a Beautiful Theorem!", a "lovely idea", a "nice", "beautiful" or even "elegant" proof.

References

ARNOLD, Vladimir I. 1997. An Interview with V. I. Arnold. *Notices of the A.M.S.* 44, 4: 432-438.

BAGLIVO, Jenny A. and Jack E. GRAVER. 1983. *Incidence and symmetry in design and architecture.* Cambridge: Cambridge University Press.

BARR, Stephen. 1964. *Experiments in topology.* New York : Crowell.

CONSIGLIERI, Luisa and Victor CONSIGLIERI. 2003. A Proposed Two-Semester Program for Mathematics in the Architecture Curriculum. *Nexus Network Journal* 5, 1: 127-134. http://www.nexusjournal.com/Didactics_v5n1-Consiglieri.html

COXETER, Harold S. M. 1961. *Introduction to Geometry.* New York: Wiley.

———. 1963. *Regular Polytopes.* New York: Macmillan.

CROWE, Donald W. 1969. Euler's formula for Polyhedra and related topics. Pp. 178 in *Excursions into Mathematics.* New York: Worth Publications.

CUNDY, H. Martyn and Arthur P. ROLLET. 1961. *Mathematical models.* Oxford : Clarendon Press.

FRANCIS, George K. 1987. *A topological picturebook.* New York : Springer.

FREUDENTHAL, Hans. 1991. *Revisiting mathematics education: China lectures.* Dordrecht: Kluwer.

GAMOV, George. 1988. *1, 2, 3 - infinity: facts and speculations of science.* New York : Dover.

GROMOV, Mikhael. 1998. Possible Trends in Mathematics on the Coming Decades. *Notices of the A.M.S.*: 45, 7: 846-847.

HAR'EL, Zvi. 1985. *Geometry for Architects.* Haifa: Technion. (in Hebrew)

———. 1993. Uniform Solution for Uniform Polyhedra. *Geometria Dedicata*, 47: 57-110.

HILBERT, David and Stephan Cohn VOSSEN. 1952. *Geometry and the imagination.* New York : Chelsea.

SOMMERVILLE, Duncan M. Y. 1958. *An introduction to the geometry of N dimensions.* New York: Dover Publications.

STEWART, Bonnie Madison. 1970. *Adventures among the toroids.* Michigan: Okemos.

TIETZE, Heinrich. 1965. *Famous problems of mathematics.* New York: Greylock Press.

THURSTON, William P. 1990. "Mathematical Education". *Notices of A.M.S.* 37, 7: 844-850.

WEIL, Hermann. 1952. *Symmetry.* Princeton, New Jersey: Princeton University Press.

WENNINGER, Magnus J. 1971. *Polyhedron models.* Cambridge: Cambridge University Press.

———. 1979. *Spherical models.* Cambridge: Cambridge University Press.

———. 1983. *Dual models.* Cambridge: Cambridge University Press.

About the author

Emil Saucan finished his BSc in mathematics at the University of Bucharest, Romania and his MSc and PhD (also in pure mathematics) at the Technion (Haifa, Israel). Formally his field was geometric function theory, but his love is geometry in general (topology included). He has taught courses in differential geometry, foundations of geometry, projective geometry (at the Technion) and discrete geometry and bio-geometric modeling at Ort Braude College, Karmiel and, of course, mathematics (geometry) for architects, which he taught, with brief interruptions, for almost two decades—first as a teaching assistant and then as a lecturer. He has written two booklets (in Hebrew): *Problems in Complex Functions* and *Problems in Topology* (with Dan Guralnik as a co-author). His interest in geometry includes its didactics and its applications in various fields, in particular in computer-aided geometric design, bio-informatics and networks.